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Practice Problem on Scattering and Phase Shifts

PHYS 852
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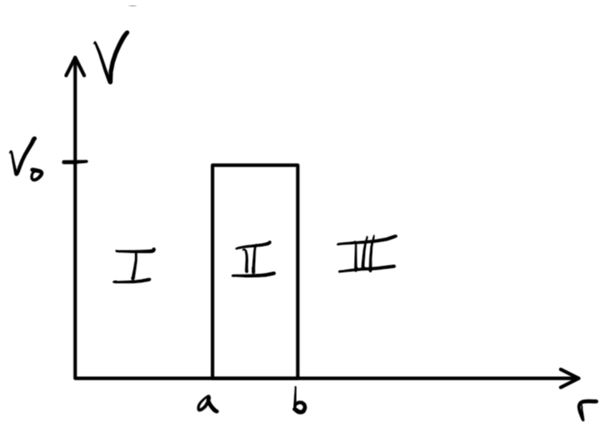
Problem

Consider the potential of what we call a soft shell:



a. Find the s-wave phase shift of a particle scattering off of a soft spherical shell of inner radius a, and outer radius b. The potential of the shell is V_0. Assume that this potential is greater than the energy of the incoming particle.

b. Check that your answer makes sense by sending a to 0 to retrieve the phase shift associated with scatting off a soft sphere. This phase shift is solved for in chapter 8 of the notes.

c. Find the cross section in the low-energy limit for the soft spherical shell.

d. Check that this answer makes sense by sending a to 0 again to retrieve the cross section of scattering off a soft sphere. This is also in the notes.

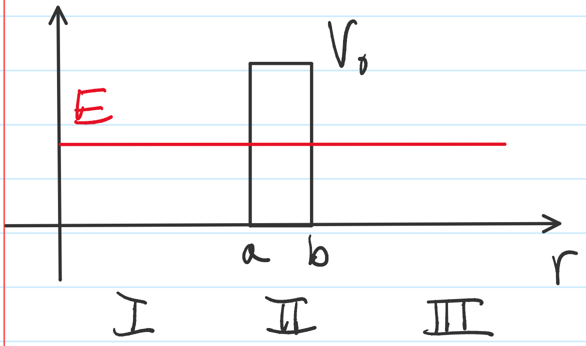
Strategy

Step 0: Freak out because this looks hard. Then take in a deep breath and tell yourself that you can do this! :) The question states that a problem like this is in the notes, so that is a good place to start. Find the problem in the notes and think about the steps involved.

Step 1: First find the wavefunction in all three regions.

Step 2: Apply boundary conditions.

Step 3: Do algebra to solve for the phase shift.



let $k=\sqrt{\frac{2mE}{\hbar^2}}$

$q=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

Step 1: Find the wave functions in each region:

As a reminder from chapter 4, the radial Schrodinger eqn is:

$$\boxed{-\frac{\hbar^2}{2m}\partial_r^2 u_\ell(r)+\left(\frac{\hbar^2\ell(\ell+1)}{2mr^2}+V(r)\right)u_\ell(r)=E\,u_\ell(r)}$$

where $u_\ell(r)\equiv rR_\ell(r)$ and $\psi(\vec{r})=R_\ell(r)Y_{\ell m}(\theta,\varphi)$

(i) The quick way to get the wave functions for s-waves

For s-waves $\ell=0$ and you can treat this exactly like a 1-D problem for $u(r)$:

$$\boxed{\begin{aligned} u_{I}(r)&=A\sin(kr)\\ u_{II}(r)&=Be^{qr}+Ce^{-qr}\\ u_{III}(r)&=\sin(kr+\delta)\end{aligned}}$$

u_I is a free particle where $u_I(0)=0$ so it must be a sin fcn

u_{III} is a free particle with a phase due to the potential

u_{II} is a sum of increasing and decaying exponentials like normal

(ii) The slow way to get the wave functions
(works for any ℓ with $V=0$ but gets ugly real quick)

The free particle solutions for any ℓ can be written as linear combinations of Henkle functions:

$$u_\ell(kr) = \frac{kr}{2}\left(\underset{\substack{\uparrow \\ \text{goes like } e^{ikr} \\ \text{at } r\to\infty}}{c_1 h_\ell(kr)} + \underset{\substack{\uparrow \\ \text{goes like } e^{-ikr} \\ \text{at } r\to\infty}}{c_2 h_\ell^*(kr)}\right)$$

where $h_\ell = j_\ell + i n_\ell$ and j_ℓ & n_ℓ are Bessel & Neumann fncs

For $\ell=0$, in region I:

$$u_I(kr) = kr\left(c_1 h_0(kr) + c_2 h_0^*(kr)\right)$$

$$= c_1\left(\sin(kr) - i\cos(kr)\right) + c_2\left(\sin(kr) + i\cos(kr)\right)$$

$$= -ic_1 e^{ikr} + ic_2 e^{-ikr} = \tilde{c}_1 e^{ikr} + \tilde{c}_2 e^{ikr}$$

$$\Rightarrow u_I(kr) = A\sin(kr) \quad \text{where } A \equiv \tilde{c}_1 = -\tilde{c}_2$$

For region III:

This is also a free particle solution like u_I but the outgoing term must be shifted by a phase due to the potential. So write this like u_I but with $\tilde{c}_1 \to \tilde{c}_1 e^{2i\delta}$

$$\Rightarrow u_{III}(kr) = \tilde{c}_1 e^{2i\delta} e^{ikr} + \tilde{c}_2 e^{ikr}$$

$$= e^{i\delta}\left(\tilde{c}_1 e^{i(kr+\delta)} + \tilde{c}_2 e^{i(kr-\delta)}\right)$$

the only physical difference between this & u_I is the δ so for simplicity we can let $\tilde{c}_1 = -\tilde{c}_2 = \text{const}$ again:

$\Rightarrow u_{III}(kr) = D\sin(kr+\delta)$

For region II:

we get $u_{II} = \tilde{C}_1 e^{ikr} + \tilde{C}_2 e^{-ikr}$ with $k \to iq$

$$= \tilde{C}_1 e^{-qr} + \tilde{C}_2 e^{qr}$$

Letting $\tilde{C}_2 = B$, $\tilde{C}_1 = C$ we get the three wave functions to be:

$$\boxed{\begin{aligned} u_I(r) &= A\sin(kr) \\ u_{II}(r) &= Be^{qr} + Ce^{-qr} \\ u_{III}(r) &= D\sin(kr+\delta) \end{aligned}}$$

The only difference between the above answer and the quick way is that A, B, C are normalized to D

Step 2: Use boundary conditions to construct system of eqns:

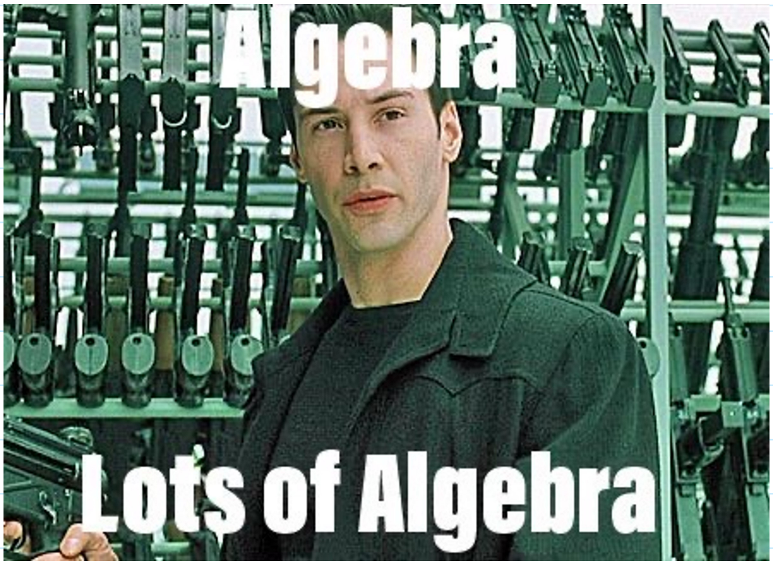
① $u_I(a) = u_{II}(a)$: $A\sin(ka) = Be^{qa} + Ce^{-qa}$

② $u_I'(a) = u_{II}'(a)$: $kA\cos(ka) = q(Be^{qa} - Ce^{-qa})$

③ $u_{III}(b) = u_{II}(b)$: $\sin(kb+\delta) = Be^{qb} + Ce^{-qb}$

④ $u_{III}'(b) = u_{II}'(b)$: $k\cos(kb+\delta) = q(Be^{qb} - Ce^{-qb})$

Step 3: Lots of Algebra



Ⓐ = ①/②: $\frac{q}{k}\tan(ka) = \frac{Be^{qa}+Ce^{-qa}}{Be^{qa}-Ce^{-qa}}$

Ⓑ = ③/④: $\frac{q}{k}\tan(kb+\delta) = \frac{Be^{qb}+Ce^{-qb}}{Be^{qb}-Ce^{-qb}}$

let $\frac{q}{k}\tan(ka)=\alpha$, $\frac{q}{k}\tan(kb+\delta) = \beta$

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rewrite Ⓐ: $\alpha = \frac{Be^{qa}+Ce^{-qa}}{Be^{qa}-Ce^{-qa}}$

$$\Rightarrow \alpha\left(Be^{qa}-Ce^{-qa}\right) = Be^{qa}+Ce^{-qa}$$

$$\Rightarrow (\alpha-1)Be^{qa} = (\alpha+1)Ce^{-qa}$$

$$\Rightarrow \frac{\alpha+1}{\alpha-1} = \frac{B}{C}e^{2qa}$$

for Ⓑ just $\alpha \to \beta$, $a \to b$: $\frac{\beta+1}{\beta-1} = \frac{B}{C}e^{2qb}$

so Ⓐ/Ⓑ: $\frac{\alpha+1}{\alpha-1}\frac{\beta-1}{\beta+1} = e^{2q(a-b)}$

let $\varphi = q(a-b)$

$$\Rightarrow \frac{\alpha\beta + (\beta-\alpha) - 1}{\alpha\beta - (\beta-\alpha) - 1} = e^{2\varphi}$$

$$\Rightarrow \alpha\beta + (\beta-\alpha) - 1 = e^{2\varphi}(\alpha\beta - (\beta-\alpha) - 1)$$

$$\Rightarrow \alpha\beta(1-e^{2\varphi}) + (\beta-\alpha)(1+e^{2\varphi}) - (1-e^{2\varphi}) = 0$$

$$\Rightarrow (\alpha\beta-1)(1-e^{2\varphi}) + (\beta-\alpha)(1+e^{2\varphi}) = 0$$

$$\Rightarrow (\alpha\beta-1)(e^{-\varphi}-e^{\varphi}) + (\beta-\alpha)(e^{-\varphi}+e^{\varphi}) = 0$$

$$\Rightarrow -(\alpha\beta-1)\sinh(\varphi) + (\beta-\alpha)\cosh(\varphi) = 0$$

$$\Rightarrow \sinh(\varphi) + \beta(\cosh(\varphi) - \alpha\sinh(\varphi)) - \alpha\cosh\varphi = 0$$

$$\Rightarrow \tanh(\varphi) - \alpha + \beta(1 - \alpha\tanh(\varphi)) = 0$$

$$\Rightarrow \beta = \frac{\alpha - \tanh(\varphi)}{1 - \alpha\tanh(\varphi)}$$

Plug in $\alpha = \frac{q}{k}\tan(ka)$, $\beta = \frac{q}{k}\tan(kb+\delta)$, $\varphi = q(a-b)$

$$\Rightarrow \frac{q}{k}\tan(kb+\delta) = \frac{\frac{q}{k}\tan(ka) - \tanh(q(a-b))}{1 - \frac{q}{k}\tan(ka)\tanh(q(a-b))}$$

a. Find phase shift

$$\boxed{\delta = -kb + \tan^{-1}\left[\frac{\tan(ka) - \frac{k}{q}\tanh(q(a-b))}{1 - \frac{q}{k}\tan(ka)\tanh(q(a-b))}\right]}$$

$$\boxed{\delta = -kb + \tan^{-1}\left[\frac{\tan(ka) - \frac{k}{q}\tanh(q(a-b))}{1 - \frac{q}{k}\tan(ka)\tanh(q(a-b))}\right]}$$

b. Send $a \to 0$ to see if it matches a soft sphere

$$\boxed{\delta_{a\to 0} = -kb + \tan^{-1}\left[\frac{k}{q}\tanh(qb)\right] \text{ which matches notes}}$$

c. Find cross section in low E limit

For low E limit $\sigma \simeq 4\pi s^2$ where s is the scattering length

$$s \equiv -\frac{\partial}{\partial k}\delta_{k=0}$$

go to linear order in k for $\delta(k)$:

$$\delta \simeq -kb + \frac{ka - \frac{k}{q}\tanh(q(a-b))}{1 - qa\tanh(q(a-b))}$$

so $$\boxed{\sigma = 4\pi s^2 \text{ where } s = b - \frac{a - \frac{1}{q}\tanh(q(a-b))}{1 - qa\tanh(q(a-b))}}$$

d. Show this matches soft sphere scattering length:

$$\boxed{s_{a\to 0} = b - \frac{1}{q}\tanh(qb) \text{ as in the notes}}$$