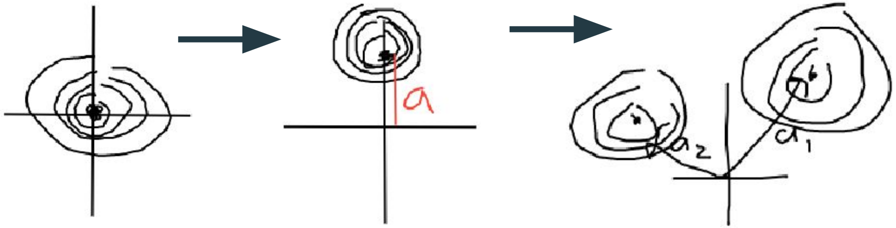
Diffraction and Forms Factors

1. Born Approximation
2. Differential Cross Section
3. Form Factor vs Structure Function

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\,\mathcal{V}(r)\,e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2$$

$$\begin{aligned} V(\vec{q}) &= \sum_a \int d^3r\,e^{i\vec{q}\cdot\vec{r}}\,V(\vec{r}-\vec{a}) \\ &= \sum_a e^{i\vec{q}\cdot\vec{a}} \int d^3r\,e^{i\vec{q}\cdot(\vec{r}-\vec{a})}\,V(\vec{r}-\vec{a}) \\ &= v(\vec{q})s(\vec{q}), \end{aligned}$$



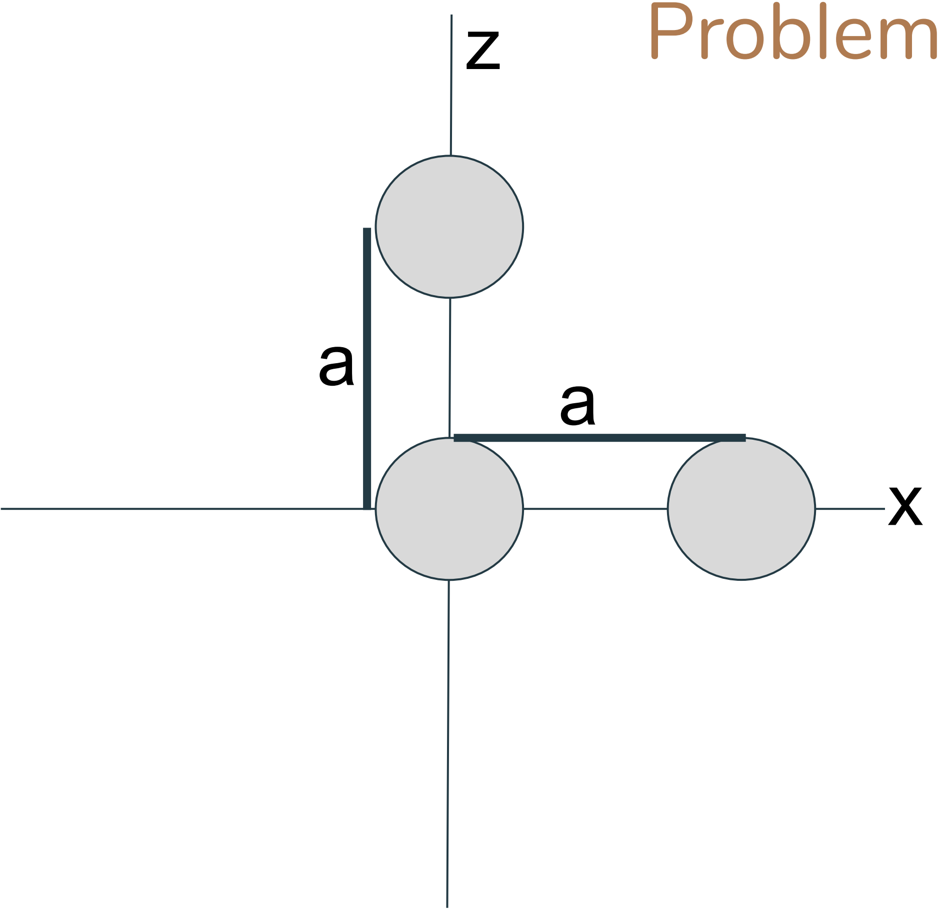
$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}} \tilde{S}(\vec{q}), \quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2$$

- When scatterers go from Discrete ⟶ Continuum, you get a Form Factor

$$|s(\vec{q})|^2 = \int d\vec{a}\,d\vec{a}'\,\rho(\vec{a})\rho(\vec{a}')e^{i\vec{q}\cdot(\vec{a}-\vec{a}')}.$$

- Form Factor: “form” for form of nucleus
- Structure Function: “structure” for structure of a crystal
- Independent vs Indistinguishable
- Interference Term ⟶ Structure Function

$$\begin{aligned} |s(\vec{q})|^2 &= \sum_{a,a'} e^{i\vec{q}\cdot(\vec{a}-\vec{a}')} \\ &= N\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}} \\ &= N\tilde{S}(\vec{q}), \\ \frac{d\sigma}{d\Omega} &= \frac{m^2}{4\pi^2\hbar^4}|v(q)|^2\tilde{S}(\vec{q}). \end{aligned}$$



Problem 7.5

The cross section of a particle with momentum ħk off of a single target is $\frac{d\sigma}{d\Omega} = \alpha,$

which is independent of **θ**. Now 2 targets are placed a distance **a** from the first target. The incoming beam is fixed to the y-z plane (***ϕ***=π/2) For what angle **θ** does the differential cross section vanish?

1

$$\tilde{S}(\vec{q}) = \left| \sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}} \right|^2$$

$$\left(1 + e^{-i\bar{q}\cdot\bar{a}_1} + e^{-i\bar{q}\cdot\bar{a}_2}\right)^2$$

$$\left(1 + e^{-iq_z a} + e^{-iq_x a}\right)^2$$

$$q_z = k_z - k_z' = k(1 - \cos\theta)$$

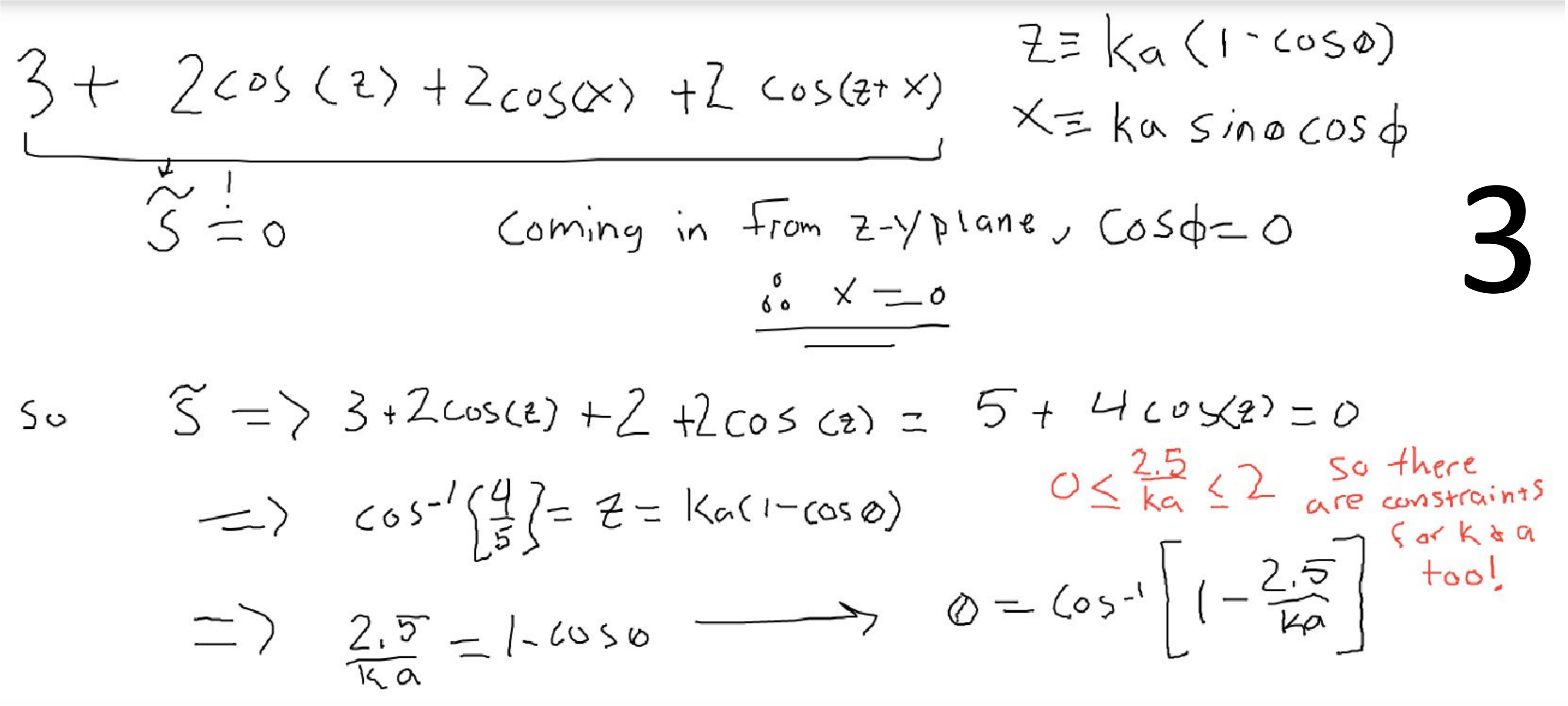
$$q_x = \overset{0}{\cancel{k_x}} - k_x' = -k\sin\theta\cos\phi$$

$$\left| 1 + e^{\overbrace{-ik(1-\cos\theta)a}^{-iz}} + e^{-ik\sin\theta\cos\phi a} \right|^2$$

$$1 + e^{z} + e^{-x} + 1 + e^{-z-x} + e^{-z} + e^{x} + e^{x+z} + 1$$

$$3 + 2\cos z + 2\cos x - 2\cos(z+x)$$

2



$$\underbrace{3 + 2\cos(z) + 2\cos(x) + 2\cos(z+x)}_{\tilde{S} \overset{!}{=} 0}$$

$z \equiv ka(1-\cos\theta)$

$x \equiv ka\sin\theta\cos\phi$

3

Coming in from z-y plane, $\cos\phi = 0$

$\therefore \; x = 0$

So $\tilde{S} \Rightarrow 3 + 2\cos(z) + 2 + 2\cos(z) = 5 + 4\cos(z) = 0$

$\Rightarrow \cos^{-1}\left\{\frac{4}{5}\right\} = z = ka(1-\cos\theta)$

$\Rightarrow \frac{2.5}{ka} = 1 - \cos\theta \longrightarrow \theta = \cos^{-1}\left[1 - \frac{2.5}{ka}\right]$

$0 \leq \frac{2.5}{ka} \leq 2$ So there are constraints for k & a too!

まっすぐまっすぐ落ちていく⇐ Forward, straightforward, I fall down.

Lecture Notes

$\boxed{1}$ Born Approx // Foundation

- lowest order perturbation theory
- incoming particle goes $\bar{k} \rightarrow \bar{k}' \equiv \bar{q}$
- single interaction w/ V @ $\bar{r}$ and its gone
- add up all the possible point interactions + square it ^{2}
- knowing this, $\frac{d\sigma}{d\Omega}$ integral is INTUITIVE (sq. fourier transf. of $V(\bar{r})$)

$\boxed{2}$ Energy Conditions

- incoming KE $\overset{!}{>}$ V
- $\bar{q}$ ~ small (unable to distinguish w/c $\overset{\text{Scat.}}{\text{center}}$ is hit)
- this is characteristic of elastic collisions

$\boxed{3}$ Origin + Translation Picture

- Origin is single scatterer Born Approx case
- if translate by $\bar{a}$, potential written $V(\bar{r}-\bar{a})$
  - pull out factor of $e^{iq\bar{a}}$ to get translated Born integral
  - separates into two terms, a $\underset{①}{\text{phase}}$ + $\underset{②}{\text{integ.}}$ (call $\underset{①}{S}$ + $\underset{②}{V}$)

$\boxed{4}$ Expand to Multiple Scattering Centers

- there are TWO centers, $\bar{a}_1$ + $\bar{a}_2$ from origin.
- thus TWO integrals + TWO phase's
  - integral dep on $\bar{r}$, not $\bar{a}$, IDENTICAL integrals
  - integral can be factored out.
- again look @ V + S terms, sq ^{2} it, AND see as $\underbrace{\underset{\text{qt.}}{\text{GEOMETRIC}} \times \overset{\text{single}}{\frac{d\sigma}{d\Omega}}}_{\frac{d\sigma}{d\Omega}\ \text{collection}}$

Geometric qt. → Structure Fct.

$\boxed{5}$ Appreciate Simplicity of the Physics

- the hard part (integral) is only done ONCE
  - reminiscent of Wigner Eckart Theorem's efficiency
- Structure given by sq.^{2} sum of exp.

$\boxed{6}$ Form Factors

- take lim $\bar{a} \rightarrow d\bar{a}$, discrete → continuous Scatterers
- location's become represented by a PROB. DENSITY, $\rho(\bar{a})$
  - $\rho(\bar{a})d\bar{a}$: chance that Scattering center is place @ $\bar{a}$
- we call the geometric qt. in Continuous lim: form factor

$\boxed{7}$ Lead (Pb) vs. Calcium (Ca)

- good example for ea. $^{5}\nearrow$ // form vs struct.
- differentiation mnemonic // Nucleus vs. Crystal
- $\overset{\text{Fac}}{\text{Form}}$ NOTE: expansion about $\bar{q} \rightarrow 0$
  - take 2ND order term
  - reminiscent of a moment! (geometric form of object)
  - just like electrostatic quadrupole tensor

$\boxed{8}$ Independent vs. Indistinguishable

- INDEP: Scattering centers far removed from one another
  - leads to vanishing cross terms ($e^{iq(a_1-a_2)}$, $e^{iq(a_2-a_3)}$, $e^{iq(a_1-a_3)}$)
  - b/c $\bar{a}_i - \bar{a}_j$ ~ BIG and $\bar{q} \cdot \delta\bar{a} < 1$
- INDIST: momentum transf. $\bar{q}$ ~ small
  - unable to tell w/c got hit, whole crystal moves, elastic

$\boxed{9}$ Interference Term (sum of all cross terms) → Structure Fct

AS WE WILL NOW SEE W/ EXAMPLE