An A particle of mass m rests in the ground state of a harmonic oscillator:

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

when it suddenly feels the effect of a harmonic potential

$$V = \alpha x \sin(\omega t)$$

with the same frequency ω as that of the oscillator. This potential allows the A particle to decay into a B particle of the same mass, which doesn't feel the harmonic oscillator potential. Calculate the decay rate for the transition.

Solution

Fermi's Golden Rule is:

$$\Gamma = \frac{2\pi}{\hbar}\sum_k |\langle f|V|i\rangle|^2 \delta(E_f - E_i).$$

The particle starts out in the lowest state of the Harmonic oscillator and ends up in a momentum eigenstate:

$$\langle x|i\rangle = \frac{1}{(\pi b^2)^{\frac{1}{4}}} e^{-\frac{x^2}{2b^2}}$$
$$\langle f|x\rangle = \frac{1}{\sqrt{L}} e^{ikx}$$

where, as usual, $b^2 = \frac{\hbar}{m\omega}$. The matrix element is of a harmonic perturbation, which has a pair of time dependent phases for decay processes and absorption processes

$$V = \alpha x \sin \omega t = \frac{\alpha x}{2i}\left(\underbrace{e^{i\omega t}}_{\text{decay}} - \underbrace{e^{-i\omega t}}_{\text{absorption}}\right).$$

Since this is a decay, the absorption piece won't contribute, so

$$\begin{aligned}
\langle k|V|n=0\rangle &= \frac{\alpha e^{i\omega t}}{2i}\frac{1}{\sqrt{L}}\frac{1}{(\pi b^2)^{\frac{1}{4}}}\int \mathrm{d}x e^{ikx} x e^{-\frac{x^2}{2b^2}} \\
&= A\int \mathrm{d}x e^{-\left(\frac{x^2}{2b^2} - ikx\right)} x \\
&= A\int \mathrm{d}x e^{-\left(\frac{x^2}{2b^2} - ikx - \frac{k^2b^2}{2} + \frac{k^2b^2}{2}\right)} x \\
&= A e^{-\frac{k^2b^2}{2}}\int \mathrm{d}x e^{-\left(\frac{x}{b\sqrt{2}} - \frac{ikb}{\sqrt{2}}\right)^2} x \\
&= Ab\sqrt{2}e^{-\frac{k^2b^2}{2}}\int \mathrm{d}u e^{-u^2}\left(\cancel{ub\sqrt{2}} + ikb^2\right)
\end{aligned}$$

where the first term was canceled through symmetry. What's left is simply

$$A\sqrt{2\pi}e^{-\frac{k^2b^2}{2}} ikb^3.$$

So the norm squared of the matrix element is

$$|M|^2 = 2\pi|A|^2 e^{-k^2b^2}k^2b^6$$

and

$$|A|^2 = \frac{\alpha^2}{4}\frac{1}{bL\sqrt{\pi}}$$

with the time dependence completely disappearing, so the norm squared is finally

$$|M|^2 = \frac{1}{2L}\left(e^{-k^2b^2}k^2b^5\alpha^2\sqrt{\pi}\right).$$

The decay rate, by Fermi's Golden Rule, is

$$\Gamma = \frac{2\pi}{\hbar}\sum_k |M|^2\delta\left(\frac{\hbar^2k^2}{2m} - \hbar\omega - \frac{1}{2}\hbar\omega\right).$$

Converting to an integral,

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar}\frac{L}{2\pi}\frac{\sqrt{\pi}b^5\alpha^2}{2L}\int \mathrm{d}k e^{-k^2b^2}k^2\delta\left(\frac{\hbar^2k^2}{2m} - \frac{3}{2}\hbar\omega\right)\\ &= \frac{\sqrt{\pi}b^5\alpha^2}{2\hbar}\frac{\hbar^2}{2m}\int \mathrm{d}k e^{-k^2b^2}k^2\delta\left(k^2 - \frac{3m\omega}{\hbar}\right)\end{aligned}$$

And the delta can be resolved using the identity

$$\delta(f(k)) = \sum_i \frac{1}{|f'(k_i)|}\delta(k - k_i)$$

for zeros of f, k_i. The zeros of f here are at $\pm\sqrt{\frac{3m\omega}{\hbar}}$ and $f'(k) = 2k$ so we get

$$\begin{aligned}\Gamma &= \frac{\sqrt{\pi}b^5\alpha^2\hbar}{4m}\sqrt{\frac{\hbar}{3m\omega}}\int \mathrm{d}k e^{-k^2b^2}k^2\delta\left(k - \sqrt{\frac{3m\omega}{\hbar}}\right)\\ &= \boxed{\frac{3\alpha^2}{\sqrt{3\pi}}\frac{b^2}{\hbar^2\omega}e^{-3}.}\end{aligned}$$