Quantum Final Presentation

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Consider the electromagnetic decay of an excited state of a 3D isotropic harmonic oscillator with quantum numbers $(n, l, m) = (0, 1, 0)$ to the ground state with $(n, l, m) = (0, 0, 0)$. In units of $\hbar = 1$, the ground state wavefunction is

$$\psi_{000} = N_{00} e^{-m\omega r^2/2} = \left(\frac{m\omega}{\pi}\right)^{3/4} e^{-m\omega r^2/2}, \tag{1}$$

and the excited state wavefunction is

$$\psi_{010}(\boldsymbol{r}) = \frac{\sqrt{2}}{\pi^{3/4}} (m\omega)^{5/4}\, r \cos\theta\, e^{-m\omega r^2/2}. \tag{2}$$

1. First, compute the differential decay rate,

$$\frac{d\Gamma}{d\Omega}, \tag{3}$$

   in the dipole approximation. Make sure to complete the sum over the polarization vectors.

2. Next, carry out the angular integral to compute the total decay rate,

$$\Gamma = \int d\Omega\, \frac{d\Gamma}{d\Omega}. \tag{4}$$

3. Finally, use the Wigner-Eckart theorem to compute the differential decay rate for $m_i = \pm 1$, and show that these give the same decay rate as that for $m_i = 0$.

Solution:

1. Let's take a moment to remember that Fermi's golden rule is:

$$\Gamma = \frac{2\pi}{\hbar}\sum_k \left|\langle f|H_{\rm int}|\psi_i\rangle\right|^2 \delta(E_0 - E_f),$$

but this can be simplified for electromagnetic decays.

Recall that the differential decay rate in the dipole approximation for electromagnetic decays[1] is

$$\frac{d\Gamma}{d\Omega} = \frac{e^2 k}{2\pi m^2}\sum_s |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2, \quad \boldsymbol{\mathcal{M}} \equiv -i\int d^3x\, \psi_f^*(\boldsymbol{x})\nabla\psi_i(\boldsymbol{x}). \tag{5}$$

Here, the final momentum k is the energy difference between the initial and final states,

$$k = E_i - E_f = \omega, \tag{6}$$

the frequency of the harmonic oscillator. Although the wavefunctions are given in spherical coordinates, the gradient is actually easier to write in Cartesian coordinates:

$$\nabla\psi_{010}(\boldsymbol{r}) = \nabla r\cos\theta e^{-m\omega r^2/2} \tag{7}$$

$$= \nabla z e^{-m\omega(x^2+y^2+z^2)/2} \tag{8}$$

$$= -m\omega e^{-m\omega(x^2+y^2+z^2)/2}\left(xz\hat{\boldsymbol{x}} + yz\hat{\boldsymbol{y}} + \left(z^2 - \frac{1}{m\omega}\right)\hat{\boldsymbol{z}}\right). \tag{9}$$

From this we can evaluate $\boldsymbol{\mathcal{M}}$:

$$i\boldsymbol{\mathcal{M}} = \int d^3x\psi_f^*\nabla\psi_i = -m\omega\left(\frac{m\omega}{\pi}\right)^{3/2}\sqrt{2m\omega}\int d^3x e^{-m\omega r^2}\left\{xz, yz, z^2 - \frac{1}{m\omega}\right\} \tag{10}$$

But the x and y components go to zero since they are odd functions over even intervals. This leaves us with:

$$i\boldsymbol{\mathcal{M}} = \int d^3x\psi_f^*\nabla\psi_i = -m\omega\left(\frac{m\omega}{\pi}\right)^{3/2}\sqrt{2m\omega}\int d^3x e^{-m\omega r^2}\left(z^2 - \frac{1}{m\omega}\right)\hat{\boldsymbol{z}} \tag{11}$$

$$= -\sqrt{\frac{2}{\pi}}(m\omega)^2\int dz e^{-m\omega z^2}\left(z^2 - \frac{1}{m\omega}\right)\hat{\boldsymbol{z}} \tag{12}$$

$$= -\sqrt{\frac{2}{\pi}}(m\omega)^2\left(-\partial_{m\omega}\int dz e^{-m\omega z^2} - \frac{\sqrt{\pi}}{(m\omega)^{3/2}}\right)\hat{\boldsymbol{z}} \tag{13}$$

$$= \sqrt{\frac{2}{\pi}}(m\omega)^2\left(\partial_{m\omega}\sqrt{\frac{\pi}{m\omega}} + \frac{\sqrt{\pi}}{(m\omega)^{3/2}}\right)\hat{\boldsymbol{z}} \tag{14}$$

$$= \sqrt{\frac{2}{\pi}}(m\omega)^2\left(\frac{\sqrt{\pi}}{(m\omega)^{3/2}} - \frac{\sqrt{\pi}}{2(m\omega)^{3/2}}\right)\hat{\boldsymbol{z}} \tag{15}$$

$$= \sqrt{\frac{m\omega}{2}}\hat{\boldsymbol{z}} \tag{16}$$

Now, we want to compute the polarization sum,

$$\sum_{s=1,2}|\boldsymbol{\epsilon}_s\cdot\boldsymbol{\mathcal{M}}|^2. \tag{17}$$

Because $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are orthonormal vectors, and are already orthogonal to the propagation vector $\hat{\boldsymbol{k}}$, these three vectors form a basis for $\mathbb{R}^3$, and we can write[2]

$$\sum_{s=1,2}|\boldsymbol{\epsilon}_s\cdot\boldsymbol{\mathcal{M}}|^2 + |\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{M}}|^2 = |\mathcal{M}|^2. \tag{18}$$

[1] Scott's Notes page 141

[2] If this feels weird, consider writing it out for the standard basis $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$, for your favorite vector.

The propagation vector is

$$\hat{\boldsymbol{k}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}. \tag{19}$$

So, for this case, the differential decay rate is

$$\frac{d\Gamma}{d\Omega} = \frac{e^2k}{2\pi m^2}\left[\sum_{s=1,2}|\boldsymbol{\epsilon}_s\cdot\boldsymbol{\mathcal{M}}|^2 + |\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{M}}|^2 - |\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{M}}|^2\right] \tag{20}$$

$$= \frac{e^2k}{2\pi m^2}\left[|\mathcal{M}|^2 - |\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{M}}|^2\right] \tag{21}$$

$$= \frac{e^2k}{2\pi m^2}\frac{m\omega}{2}\left[1-\cos^2\theta\right] = \boxed{\frac{e^2\omega^2}{4\pi m}\sin^2\theta.} \tag{22}$$

2. To get the Total decay rate we simply integrate over the angular distribution:

$$\Gamma = \frac{e^2\omega^2}{4\pi m}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin^3(\theta) \tag{23}$$

$$= \frac{2e^2\omega^2}{3m} \tag{24}$$

3. To use the Wigner-Eckart theorem, write the matrix elements as

$$\langle nlm|P_i|n'l'm'\rangle, \tag{25}$$

and recall that we can write the momentum operator as a tensor operator of rank 1 as follows:

$$P_0 = P_z, \quad P_{\pm 1} = \mp\frac{P_x \pm iP_y}{\sqrt{2}}. \tag{26}$$

Inverting the latter lets us write

$$P_x = \frac{P_{-1} - P_1}{\sqrt{2}}, \quad P_y = \frac{i(P_{-1}+P_1)}{\sqrt{2}}. \tag{27}$$

Then, the Wigner-Eckart theorem lets us write the matrix elements as

$$\langle nlm|P_\mu|n'l'm'\rangle = \langle nl||P||n'l'\rangle\langle l',m';1,\mu|l,m\rangle = -i\sqrt{\frac{m\omega}{2}}\frac{\langle l',m';1,\mu|l,m\rangle}{\langle 1,0;1,0|0,0\rangle}. \tag{28}$$

So, we see that

$$\langle\psi_{000}|\boldsymbol{P}|\psi_{01\mu}\rangle = \langle\psi_{000}|P_x|\psi_{01\mu}\rangle\hat{\boldsymbol{x}} + \langle\psi_{000}|P_y|\psi_{01\mu}\rangle\hat{\boldsymbol{y}} + \langle\psi_{000}|P_z|\psi_{01\mu}\rangle\hat{\boldsymbol{z}} \tag{29}$$

$$= \frac{1}{\sqrt{2}}\langle\psi_{000}|P_{-1}|\psi_{01\mu}\rangle\left[\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right] + \frac{1}{\sqrt{2}}\langle\psi_{000}|P_1|\psi_{01\mu}\rangle\left[-\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right] + \langle\psi_{000}|P_0|\psi_{01\mu}\rangle\hat{\boldsymbol{z}} \tag{30}$$

$$= \frac{1}{\sqrt{2}}(-i)\sqrt{\frac{m\omega}{2}}\frac{\langle 1,\mu;1,-1|0,0\rangle}{\langle 1,0;1,0|0,0\rangle}\left[\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right] + \frac{1}{\sqrt{2}}(-i)\sqrt{\frac{m\omega}{2}}\frac{\langle 1,\mu;1,1|0,0\rangle}{\langle 1,0;1,0|0,0\rangle}\left[-\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right]$$
$$(-i)\sqrt{\frac{m\omega}{2}}\frac{\langle 1,\mu;1,0|0,0\rangle}{\langle 1,0;1,0|0,0\rangle}\hat{\boldsymbol{z}} \tag{31}$$

$$= i\frac{\sqrt{m\omega}}{2}\left[\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right]\delta_{\mu 1} + \frac{\sqrt{m\omega}}{2}\left[-\hat{\boldsymbol{x}}+i\hat{\boldsymbol{y}}\right]\delta_{\mu,-1} - i\sqrt{\frac{m\omega}{2}}\hat{\boldsymbol{z}}\delta_{\mu 0}. \tag{32}$$

Now, the differential decay rate can be evaluated using the same trick as the case where $\mu = 0$. For $\mu = 1$, this gives

$$\frac{d\Gamma_{\mu=1}}{d\Omega} = \frac{e^2\omega}{2\pi m^2}\left[|\mathcal{M}|^2 - |\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{M}}|^2\right] \tag{33}$$

$$= \frac{e^2\omega}{2\pi m^2}\left[\frac{m\omega}{2} - \frac{m\omega}{4}|\sin\theta\cos\phi + i\sin\theta\sin\phi|^2\right] \tag{34}$$

$$= \boxed{\frac{e^2\omega^2}{8\pi m}\left[2 - \sin^2\theta\right].} \tag{35}$$

Note that this is the same result we get for $\mu = -1$. Integrating this gives the total decay rate:

$$\Gamma_{\mu=\pm 1} = 2\pi \frac{e^2\omega^2}{8\pi m} \int_0^\pi d\theta \left[2\sin\theta - \sin^3\theta\right] \tag{36}$$

$$= \frac{e^2\omega^2}{4m} \cdot \frac{8}{3} = \boxed{\frac{2}{3}\frac{e^2\omega^2}{m}.} \tag{37}$$

Note that this decay rate is the same as $\mu = 0$, as expected.