Quantum Final Presentation

Chapter 9: Decays

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Outline

The Problem

Consider the electromagnetic decay of an excited state of a 3D isotropic harmonic oscillator with quantum numbers

$$(n_i, l_i, m_i) = (1, 1, 0) \tag{1.1}$$

to the ground state with quantum numbers

$$(n_f, l_f, m_f) = (0, 0, 0) \tag{1.2}$$

The wavefunctions are

$$\psi_{000}(\boldsymbol{r}) = \left(\frac{m\omega}{\pi}\right)^{3/4} e^{-m\omega r^2/2}, \tag{1.3}$$

$$\psi_{110}(\boldsymbol{r}) = \frac{\sqrt{2}}{\pi^{3/4}}(m\omega)^{5/4}\, r\cos\theta\, e^{-m\omega r^2/2}. \tag{1.4}$$

The Problem - Part 1

First, compute the differential decay rate,

$$\frac{d\Gamma}{d\Omega}, \tag{1.5}$$

in the dipole approximation. Make sure to complete the sum over the polarization vectors.

The Problem - Part 2

Next, carry out the angular integral to compute the total decay rate,

$$\Gamma = \int d\Omega \, \frac{d\Gamma}{d\Omega}. \tag{1.6}$$

The Problem - Part 3

Finally, use the Wigner-Eckart theorem to compute the differential decay rate for $m_i = \pm 1$, and show that these give the same decay rate as that for $m_i = 0$.

Conceptual Goals

- Fermi's golden rule
- Dipole approximation
- Integrating things that look like Gaussians
- Polarization sums
- Wigner-Eckart theorem

The Solution - Part 1

Fermi's golden rule gives the differential decay rate. It is

$$\frac{d\Gamma}{d\Omega} = \frac{e^2 k}{2\pi m^2} \sum_s |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2, \tag{3.1}$$

$$\boldsymbol{\mathcal{M}} \equiv -i \int d^3 r\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \psi_f^*(\boldsymbol{x}) \nabla \psi_i(\boldsymbol{x}), \tag{3.2}$$

$$k = E_i - E_f = \omega. \tag{3.3}$$

In $\mathcal{M}$, the approximation $e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \approx 1$ is the dipole approximation.

The Solution - Part 1

$$\boldsymbol{\mathcal{M}} \equiv -i \int d^3r \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \psi_f^*(\boldsymbol{x}) \nabla \psi_i(\boldsymbol{x})$$

Re-writing the wave functions in Cartesian, the gradient is straightforward to compute. The matrix element is

$$\boldsymbol{\mathcal{M}} = -\frac{\sqrt{2}(m\omega)^3}{\pi^{3/2}} \int d^3r \, e^{-m\omega r^2} \left(z^2 - \frac{1}{m\omega} \right) \hat{\boldsymbol{z}} \tag{3.4}$$

The Solution - Part 1

For the z^2 term of the $\boldsymbol{\mathcal{M}}$ we compute the integral as:

$$-\int_{-\infty}^{\infty} dz\, z^2 e^{-m\omega z^2} = \frac{\partial}{\partial(m\omega)} \int_{-\infty}^{\infty} dz\, e^{-m\omega z^2} \tag{3.5}$$

$$= \frac{\partial}{\partial(m\omega)} \sqrt{\frac{\pi}{m\omega}} = -\frac{1}{2} \frac{\sqrt{\pi}}{(m\omega)^{3/2}}. \tag{3.6}$$

This method is called differentiating under the integral.

The Solution - Part 1

The matrix element simplifies to

$$\boldsymbol{\mathcal{M}} = -i\sqrt{\frac{m\omega}{2}}\hat{\boldsymbol{z}}. \tag{3.7}$$

which we plug into

$$\frac{d\Gamma}{d\Omega} = \frac{e^2 k}{2\pi m^2} \sum_s |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2$$

We want to compute the polarization sum

$$\sum_{s=1,2} |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2. \tag{3.8}$$

The Solution - Part 1

The two polarization vectors and the propagation vectors

$$(\boldsymbol{\epsilon}_1,\, \boldsymbol{\epsilon}_2,\, \hat{\boldsymbol{k}}) \tag{3.9}$$

form an orthonormal basis for $\mathbb{R}^3$, so

$$\sum_{s=1,2} |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2 + |\hat{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{M}}|^2 = |\mathcal{M}|^2. \tag{3.10}$$

This lets us write

$$\sum_{s=1,2} |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2 = \sum_{s=1,2} |\boldsymbol{\epsilon}_s \cdot \boldsymbol{\mathcal{M}}|^2 + |\hat{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{M}}|^2 - |\hat{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{M}}|^2 \tag{3.11}$$

$$= |\mathcal{M}|^2 - |\hat{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{M}}|^2. \tag{3.12}$$

Polarization Basis Visual

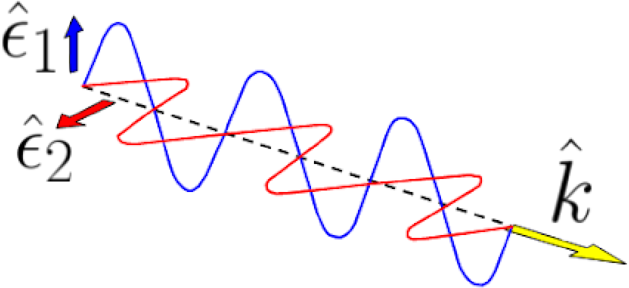

Figure 1: Image of propagation and polarization vectors.

The Solution - Part 1

The propagation vector can be written as

$$\hat{\boldsymbol{k}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}. \tag{3.13}$$

So, we conclude that

$$\boxed{\frac{d\Gamma}{d\Omega} = \frac{e^2\omega^2}{4\pi m}\sin^2\theta.} \tag{3.14}$$

The Problem - Part 2

Next, carry out the angular integral to compute the total decay rate,

$$\Gamma = \int d\Omega \, \frac{d\Gamma}{d\Omega}. \tag{4.1}$$

The Solution - Part 2

The total decay rate is

$$\Gamma = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \frac{d\Gamma}{d\Omega} \tag{4.2}$$

$$= \boxed{\frac{2}{3}\frac{e^2\omega^2}{m}.} \tag{4.3}$$

The Problem - Part 3

Finally, use the Wigner-Eckart theorem to compute the differential decay rate for $m_i = \pm 1$, and show that these give the same decay rate as that for $m_i = 0$.

The Solution - Part 3

To use the Wigner-Eckart theorem, write the matrix elements as

$$\mathcal{M}_i = \langle nlm|P_i|n'l'm'\rangle, \tag{5.1}$$

and recall the spherical tensors

$$P_0 = P_z, \quad P_{\pm 1} = \mp \frac{P_x \pm iP_y}{\sqrt{2}}. \tag{5.2}$$

The Solution - Part 3

The Wigner-Eckart theorem lets us write the matrix elements as

$$\langle nl\mu|P_q|n'l'\mu'\rangle = \langle nl||P||n'l'\rangle\langle l',\mu';1,q|l,\mu\rangle \tag{5.3}$$

$$= -i\sqrt{\frac{m\omega}{2}}\frac{\langle l',\mu';1,q|l,\mu\rangle}{\langle 1,0;1,0|0,0\rangle}, \tag{5.4}$$

for $q = 0, \pm 1$. We can write

$$\langle 000|\boldsymbol{P}|01m_i\rangle = \langle 000|P_x|01m_i\rangle\hat{\boldsymbol{x}} + \langle 000|P_y|01m_i\rangle\hat{\boldsymbol{y}} + \langle 000|P_z|01m_i\rangle\hat{\boldsymbol{z}}. \tag{5.5}$$

The Solution - Part 3

In our spherical tensors, solving for P_x, P_y, P_z in terms of $P_{0,\pm 1}$ lets us write

$$\begin{aligned}\boldsymbol{\mathcal{M}} = \langle 000|\boldsymbol{P}|01m_i\rangle = \frac{1}{\sqrt{2}}\langle 000|P_{-1}|01m_i\rangle\big[\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}\big] \\ + \frac{1}{\sqrt{2}}\langle 000|P_{1}|01m_i\rangle\big[-\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}\big] \\ + \langle 000|P_0|01m_i\rangle\hat{\boldsymbol{z}}. \end{aligned} \tag{5.6}$$

Using the Wigner-Eckart theorem gives us

$$\begin{aligned}\boldsymbol{\mathcal{M}} = i\frac{\sqrt{m\omega}}{2}\big[\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}\big]\delta_{m_i 1} + i\frac{\sqrt{m\omega}}{2}\big[-\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}\big]\delta_{m_i,-1} \\ - i\sqrt{\frac{m\omega}{2}}\hat{\boldsymbol{z}}\delta_{m_i 0}. \end{aligned} \tag{5.7}$$

The Solution - Part 3

Evaluating the polarization sum in the same way as the $\mu = 0$ case gives

$$\frac{d\Gamma_{m_i=\pm 1}}{d\Omega} = \frac{e^2\omega}{2\pi m^2}\left[|\mathcal{M}|^2 - |\hat{\boldsymbol{k}} \cdot \boldsymbol{\mathcal{M}}|^2\right] \tag{5.8}$$

$$= \boxed{\frac{e^2\omega^2}{8\pi m}\left[2 - \sin^2\theta\right].} \tag{5.9}$$

Integrating directly shows that

$$\Gamma_{m_i=\pm 1} = \Gamma_{m_i=0} = \frac{2}{3}\frac{e^2\omega^2}{m}. \tag{5.10}$$