Problem

a)
Construct the matrix representations of $S_y$ and $S_z$ for a spin-1 particle using the basis

$|m = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|m = 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|m = -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

What is the behavior of $S_\pm$ under time reversal?

b)
Given the Hamiltonian

$H = AS_x^2 + B(S_x^2 - S_y^2)$

with $A, B \in \mathbb{R}$, find the eigenstates and energies. How does the Hamiltonian behave under time reversal? What about the eigenstates and energies?

Solutions

a)
By construction, $S_z$ is diagonal in this basis. We have

$S_z = \hbar |1\rangle \langle 1| + 0 |0\rangle \langle 0| - \hbar |-1\rangle \langle -1| = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Now, to build $S_y$, we let’s use operators that we already know the behavior of:

$S_\pm = S_x \pm iS_y$.

This tells us

$S_y = \frac{1}{2i}(S_+ - S_-)$,

so

$\langle m'|S_y|m\rangle = \frac{1}{2i} \langle m'|S_+ - S_- |m\rangle$

$= \frac{\hbar}{2i} \left( \sqrt{(1-m)(2+m)}\delta_{m',m+1} - \sqrt{(1+m)(2-m)}\delta_{m',m-1} \right)$

$\implies S_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.
after calculating out these matrix elements. While not asked by the problem, it will be useful to know later that a similar calculation using \( S_x = \frac{1}{2}(S_+ + S_-) \) yields

\[
S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

Now, how do \( S_\pm \) behave under time reversal? Well, we know on physical grounds that spin operators transform as

\[
\Theta S_i \Theta^{-1} = -S_i.
\]

We also know that \( \Theta \) acts as an antiunitary operator, i.e. it conjugates coefficients as it passes through them. Thus,

\[
\Theta S_\pm \Theta^{-1} = (\Theta S_x \mp i\Theta S_y)\Theta^{-1} = -(S_x \pm iS_y) = -S_\mp.
\]

b)

Using our computed matrix representations, the Hamiltonian has the matrix

\[
H = AS_x^2 + B(S_x^2 - S_y^2)
\]

\[
= \hbar^2 \left[ A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 + \frac{B}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 + \frac{B}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 \right]
\]

\[
= \hbar^2 \left[ A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{B}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{B}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \right]
\]

\[
= \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}
\]

The eigensystem of this matrix is easily solved in the standard way, yielding

\[
E_+ = \hbar^2(A + B); \quad |E_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]

\[
E_- = \hbar^2(A - B); \quad |E_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\]

\[
E_0 = 0; \quad |E_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]
How does $H$ act under time reversal? We have

$$\Theta H \Theta^{-1} = \Theta \left( A S_z^2 + B (S_x^2 - S_y^2) \right) \Theta^{-1}$$

$$= A \Theta S_z^2 \Theta^{-1} + B (\Theta S_x^2 \Theta^{-1} - \Theta S_y^2 \Theta^{-1})$$

$$= A(\Theta S_z \Theta^{-1})(\Theta S_z \Theta^{-1}) + B \left( (\Theta S_x \Theta^{-1})(\Theta S_z \Theta^{-1}) - (\Theta S_y \Theta^{-1})(\Theta S_y \Theta^{-1}) \right)$$

$$= AS_z^2 + B (S_x^2 - S_y^2).$$

So, $H$ is invariant under time reversal.

Now, how do the eigenstates transform under time reversal? We know

$$\Theta |j, m\rangle = (-1)^m |j, -m\rangle,$$

so

$$\Theta |E_+\rangle = \frac{1}{\sqrt{2}} \Theta \left( |m = 1\rangle + |m = -1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( (-1) |m = -1\rangle + (-1) |m = 1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} (-1) \left( |m = -1\rangle + |m = 1\rangle \right)$$

$$= - |E_+\rangle$$

$$\Theta |E_-\rangle = \frac{1}{\sqrt{2}} \Theta \left( |m = 1\rangle - |m = -1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( (-1) |m = -1\rangle - (-1) |m = 1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( |m = 1\rangle - |m = -1\rangle \right)$$

$$= |E_-\rangle$$

$$\Theta |E_0\rangle = \Theta |m = 0\rangle$$

$$= |m = 0\rangle$$

$$= |E_0\rangle$$

So, the $|E_+\rangle$ eigenstate is odd under time reversal while the others are even. Notice that the time reversal simply introduces a phase to the state, and thus does not change the energy.