PHY 852 Review: Time Reversal

Tyler Weitzel, Hieu Le

April 13, 2020

Problem

a)

Construct the matrix representations of S_y and S_z for a spin-1 particle using the basis

$$|m=1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \; |m=0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \; |m=-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

What is the behavior of $S_\pm$ under time reversal?

b)

Given the Hamiltonian

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

with $A, B \in \mathbb{R}$, find the eigenstates and energies. How does the Hamiltonian behave under time reversal? What about the eigenstates and energies?

Solutions

a)

By construction, S_z is diagonal in this basis. We have

$$S_z = \hbar\,|1\rangle\langle 1| + 0\,|0\rangle\langle 0| - \hbar\,|-1\rangle\langle -1| = \hbar \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Now, to build S_y, we let's use operators that we already know the behavior of:

$$S_\pm = S_x \pm iS_y.$$

This tells us

$$S_y = \frac{1}{2i}(S_+ - S_-),$$

so

$$\begin{aligned}
\langle m'|\,S_y\,|m\rangle &= \frac{1}{2i}\,\langle m'|\,S_+ - S_-\,|m\rangle \\
&= \frac{\hbar}{2i}\left(\sqrt{(1-m)(2+m)}\,\delta_{m',m+1} - \sqrt{(1+m)(2-m)}\,\delta_{m',m-1}\right) \\
\implies S_y &= \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}
\end{aligned}$$

after calculating out these matrix elements. While not asked by the problem, it will be useful to know later that a similar calculation using $S_x = \frac{1}{2}(S_+ + S_-)$ yields

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now, how do $S_\pm$ behave under time reversal? Well, we know on physical grounds that spin operators transform as

$$\Theta S_i \Theta^{-1} = -S_i.$$

We also know that Θ acts as an antiunitary operator, i.e. it conjugates coefficients as it passes through them. Thus,

$$\begin{aligned} \Theta S_\pm \Theta^{-1} &= \Theta(S_x \pm iS_y)\Theta^{-1} \\ &= (\Theta S_x \mp i\Theta S_y)\Theta^{-1} \\ &= (\Theta S_x \Theta^{-1}) \mp i(\Theta S_y \Theta^{-1}) \\ &= -S_x \pm iS_y \\ &= -(S_x \mp iS_y) \\ &= \boxed{-S_\mp}. \end{aligned}$$

b)

Using our computed matrix representations, the Hamiltonian has the matrix

$$\begin{aligned} H &= AS_z^2 + B(S_x^2 - S_y^2) \\ &= \hbar^2 \Big[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 + \frac{B}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 + \frac{B}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 \Big] \\ &= \hbar^2 \Big[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{B}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{B}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Big] \\ &= \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix} \end{aligned}$$

The eigensystem of this matrix is easily solved in the standard way, yielding

$$\begin{aligned} E_+ &= \hbar^2(A+B);\ |E_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ E_- &= \hbar^2(A-B);\ |E_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ E_0 &= 0;\ |E_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

How does H act under time reversal? We have

$$\begin{aligned}
\Theta H \Theta^{-1} &= \Theta\Big(AS_z^2 + B(S_x^2 - S_y^2)\Big)\Theta^{-1} \\
&= A\Theta S_z^2\Theta^{-1} + B(\Theta S_x^2\Theta^{-1} - \Theta S_y^2\Theta^{-1}) \\
&= A(\Theta S_z\Theta^{-1})(\Theta S_z\Theta^{-1}) + B\Big((\Theta S_x\Theta^{-1})(\Theta S_x\Theta^{-1}) - (\Theta S_y\Theta^{-1})(\Theta S_y\Theta^{-1}\Big) \\
&= AS_z^2 + B(S_x^2 - S_y^2).
\end{aligned}$$

So, H is invariant under time reversal.

Now, how do the eigenstates transform under time reversal? We know

$$\Theta\,|j,m\rangle = (-1)^m\,|j,-m\rangle\,,$$

so

$$\begin{aligned}
\Theta\,|E_+\rangle &= \frac{1}{\sqrt{2}}\Theta\Big(\,|m=1\rangle + |m=-1\rangle\,\Big) \\
&= \frac{1}{\sqrt{2}}\Big((-1)\,|m=-1\rangle + (-1)\,|m=1\rangle\,\Big) \\
&= \frac{1}{\sqrt{2}}(-1)\Big(\,|m=-1\rangle + |m=1\rangle\,\Big) \\
&= -\,|E_+\rangle \\
\Theta\,|E_-\rangle &= \frac{1}{\sqrt{2}}\Theta\Big(\,|m=1\rangle - |m=-1\rangle\,\Big) \\
&= \frac{1}{\sqrt{2}}\Big((-1)\,|m=-1\rangle - (-1)\,|m=1\rangle\,\Big) \\
&= \frac{1}{\sqrt{2}}\Big(\,|m=1\rangle - |m=-1\rangle\,\Big) \\
&= |E_-\rangle \\
\Theta\,|E_0\rangle &= \Theta\,|m=0\rangle \\
&= |m=0\rangle \\
&= |E_0\rangle
\end{aligned}$$

So, the $|E_+\rangle$ eigenstate is odd under time reversal while the others are even. Notice that the time reversal simply introduces a phase to the state, and thus does not change the energy.