Chapter 3: Charged Particles in an Electromagnetic Field

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Spring 2020

The basics of interactions with some background electromagnetic field in quantum mechanics are as follows.

- All of this is in Gaussian units where $\mathbf{E}$ and $\mathbf{B}$ have the same units, hence weird factors of c
- Minimal coupling: $\mathbf{p} \to \mathbf{\Pi} = \mathbf{p} - \dfrac{e\mathbf{A}}{c}$ and $E \to E - e\Phi$
- The canonical momentum $\mathbf{\Pi}$ satisfies the relation $\mathbf{\Pi} = m\mathbf{v}$
- Schrödinger equation becomes $\left[\dfrac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m} + V(\mathbf{r}) + e\Phi\right]\psi = E\psi$
- Classical results are reproduced with these modifications, e.g. helical motion in criss-crossed $\mathbf{E}$ and $\mathbf{B}$ fields and cyclotron frequency

Problem

A particle of mass m and electric charge e is placed in the uniform static magnetic field $\mathbf{B} = B\hat{z}$ and the harmonic oscillator potential along the x-axis (frequency ω_x) and along the z-axis (frequency ω_z) Find the energy spectrum and the wavefunctions of the stationary states.

Solution

First we need to pick a vector potential that will produce this magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Because of the harmonic oscillator in x, it is easiest if we pick $\mathbf{A}$ to be only in the y-direction. The vector potential

$$\mathbf{A} = Bx\hat{y} \tag{1}$$

will produce the required magnetic field. Then the Schrödinger equation is

$$\left[\frac{p_x^2}{2m} + \frac{(p_y - eBx/c)^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_x^2x^2 + \frac{1}{2}m\omega_z^2z^2\right]\psi = E\psi \tag{2}$$

We can use separation of variables to rewrite the wavefunction as

$$\psi(x,y,z) = X(x)Y(y)Z(z) \tag{3}$$

Then we can solve the Schrödinger equation for each coordinate separately. The y direction has a free particle, so its wavefunction is just

$$Y(y) = e^{ik_y y} \tag{4}$$

The x-direction is not a free particle since the coordinate x appears in the Hamiltonian through the vector potential. If you do the separation of variables, you get the following equations for X and Z.

$$\left(\frac{p_x^2}{2m} + \frac{(p_y - eBx/c)^2}{2m} + \frac{1}{2}m\omega_x^2 x^2\right) X(x) = E_{xy}X(x) \tag{5}$$

$$\left(\frac{p_z^2}{2m} + \frac{1}{2}m\omega_z^2 z^2\right) Z(z) = E_z Z(z) \tag{6}$$

$$E = E_{xy} + E_z \tag{7}$$

In the $X(x)$ equation, we can keep p_y as p_y since it is a constant of motion. That is, p_y commutes with the Hamiltonian since there are no y terms. We don't need to replace it with $-i\hbar\partial_y$ since that would simply act on $Y(y)$ and pull out its eigenvalue, which is p_y.

The $Z(z)$ equation is easy to solve since it is a harmonic oscillator in the z-direction centered at $z = 0$. Thus

$$E_z = \hbar\omega_z\left(n_z + \frac{1}{2}\right) \tag{8}$$

The $X(x)$ equation is a little trickier. We need to rewrite the "potential" terms to put them in a more useful form.

$$\begin{aligned}
\frac{(p_y - eBx/c)^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 &= \frac{(eB/c)^2(x - cp_y/eB)^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 \\
&= \frac{1}{2}m\left(\frac{eB}{mc}\right)^2\left(x - \frac{cp_y}{eB}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 \\
&= \frac{1}{2}m\omega_c^2(x - x_0)^2 + \frac{1}{2}m\omega_x^2 x^2 \\
\omega_c \equiv \frac{eB}{mc} &\quad x_0 \equiv \frac{cp_y}{eB}
\end{aligned}$$

Note that ω_c is the cyclotron frequency, which is a classical result that has been reproduced. Now we have two oscillator potentials, but one is centered at $x = x_0$ and the other is centered at $x = 0$. We need to rewrite the terms to write the potential as a single oscillator and constant terms. This involves some messy algebra and completing the square. All the steps will be shown below, but if you just want the final result, then look at the first and last lines.

$$\begin{aligned}
\frac{1}{2}m\omega_c^2(x-x_0)^2+\frac{1}{2}m\omega_x^2x^2 &= \frac{1}{2}m(\omega_c^2x^2-2\omega_c^2x_0x+\omega_c^2x_0^2+\omega_x^2x^2)\\
&= \frac{1}{2}m\left[(\omega_c^2+\omega_x^2)x^2-2\omega_c^2x_0x+\omega_c^2x_0^2\right]\\
&= \frac{1}{2}m(\omega_c^2+\omega_x^2)\left(x^2-\frac{2\omega_c^2x_0}{\omega_c^2+\omega_x^2}x+\frac{\omega_c^2x_0^2}{\omega_c^2+\omega_x^2}\right)\\
&= \frac{1}{2}m\Omega^2\left(x^2-\frac{2\omega_c^2x_0}{\Omega^2}x+\frac{\omega_c^2x_0^2}{\Omega^2}\right)
\end{aligned}$$

We define a new frequency $\Omega^2\equiv\omega_c^2+\omega_x^2$ and complete the square.

$$\begin{aligned}
\frac{1}{2}m\Omega^2\left(x^2-\frac{2\omega_c^2x_0}{\Omega^2}x+\frac{\omega_c^2x_0^2}{\Omega^2}\right) &= \frac{1}{2}m\Omega^2\left(x^2-\frac{2\omega_c^2x_0}{\Omega^2}x\right)+\frac{1}{2}m\omega_c^2x_0^2\\
&= \frac{1}{2}m\Omega^2\left(x^2-\frac{2\omega_c^2x_0}{\Omega^2}x+\frac{\omega_c^4x_0^2}{\Omega^4}\right)+\frac{1}{2}m\omega_c^2x_0^2-\frac{1}{2}m\frac{\omega_c^4x_0^2}{\Omega^2}\\
&= \frac{1}{2}m\Omega^2\left(x-\frac{\omega_c^2x_0}{\Omega^2}\right)^2+\frac{1}{2}m\omega_c^2x_0^2\left(1-\frac{\omega_c^2}{\Omega^2}\right)\\
&= \frac{1}{2}m\Omega^2\left(x-\frac{\omega_c^2x_0}{\Omega^2}\right)^2+\frac{1}{2}m\omega_c^2x_0^2\left(1-\frac{\omega_c^2}{\omega_c^2+\omega_x^2}\right)\\
&= \frac{1}{2}m\Omega^2\left(x-\frac{\omega_c^2x_0}{\Omega^2}\right)^2+\frac{1}{2}m\omega_c^2x_0^2\left(\frac{\omega_x^2}{\omega_c^2+\omega_x^2}\right)\\
&= \frac{1}{2}m\Omega^2\left(x-\frac{\omega_c^2x_0}{\Omega^2}\right)^2+\frac{1}{2}m\frac{\omega_x^2\omega_c^2}{\Omega^2}x_0^2
\end{aligned}$$

If we use the fact that $\omega_cx_0=p_y/m$ from how those variables were defined, we can rewrite the potential term as

$$\frac{1}{2}m\Omega^2\left(x-\frac{\omega_cp_y}{m\Omega^2}\right)^2+\frac{\omega_x^2p_y^2}{2m\Omega^2} \tag{9}$$

Thus, the Schrödinger equation for $X(x)$ becomes

$$\left[\frac{p_x^2}{2m}+\frac{1}{2}m\Omega^2\left(x-\frac{\omega_cp_y}{m\Omega^2}\right)^2+\frac{\omega_x^2p_y^2}{2m\Omega^2}\right]X(x)=E_{xy}X(x) \tag{10}$$

After making the substitution $p_x=-i\hbar\partial_x$ and distributing $X(x)$, this becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\Omega^2\left(x-\frac{\omega_cp_y}{m\Omega^2}\right)^2\right]X(x)+\frac{\omega_x^2p_y^2}{2m\Omega^2}X(x)=E_{xy}X(x) \tag{11}$$

The first term is a harmonic oscillator potential with frequency $\Omega=\sqrt{\omega_c^2+\omega_x^2}$ and centered at $x=\frac{\omega_cp_y}{m\Omega^2}$. Therefore, we can simply replace that part of the Hamiltonian with the eigenenergies of that Hamiltonian. Then the partial derivative goes away and we are just left with numbers instead of operators.

$$\hbar\Omega\left(n+\frac{1}{2}\right)X(x)+\frac{\omega_x^2p_y^2}{2m\Omega^2}X(x)=E_{xy}X(x) \tag{12}$$

Now since everything is in terms of numbers are not operators, we are free to divide out $X(x)$ and get the energy.

$$E_{xy} = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\omega_x^2 p_y^2}{2m\Omega^2} \tag{13}$$

Thus the energy spectrum is

$$\boxed{E = E_{xy} + E_z = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\omega_x^2 p_y^2}{2m\Omega^2} + \hbar\omega_z\left(n_z + \frac{1}{2}\right) \quad n, n_z \in \mathbb{N}} \tag{14}$$

The wavefunction for $Y(y)$ has already been explained to be that of a free particle. For $Z(z)$, it is a regular 1D harmonic oscillator centered at $z = 0$. For $X(x)$, the potential looks like a harmonic oscillator except there is an additional term $\dfrac{\omega_x^2 p_y^2}{2m\Omega^2}$. However, this term is a constant, so it only contributes a phase to the wavefunction, which we will ignore, so that $X(x)$ is a 1D oscillator as well.

$$\boxed{\psi(x,y,z) = \phi_n\left(x - \frac{\omega_c p_y}{m\Omega^2}, \text{oscillator frequency} = \Omega\right)\chi_{n_z}\left(z, \text{oscillator frequency} = \omega_z\right) e^{ik_y y}} \tag{15}$$

Additional Remarks

- Notice that the additional oscillator in x has lifted the degeneracy for different p_y since the E_{xy} energy would otherwise be independent of p_y. With the additional oscillator, the energy now depends on p_y. Otherwise, p_y does little more than add a phase to the wavefunction and contribute to the shift in the x oscillator.
- With only a magnetic field, the oscillator is shifted by $x_0 = \dfrac{cp_y}{eB}$. Here the shift is $x_0 = \dfrac{\omega_c p_y}{m\Omega^2}$. Clearly, this is a different shift.
- The interaction between $\mathbf{B}$ and ω_x: in addition to lifting the degeneracy from p_y, there is some interesting interaction between the magnetic field and the x oscillator. We still have an oscillator, but its frequency is the sum of the cyclotron and regular oscillator frequencies in quadrature. We also get a new constant term that depends on the frequencies.