Chapter 13: Relativistic Quantum Mechanics

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The problem

Suppose that an electron of momentum k coming from the left strikes a one-dimensional potential barrier

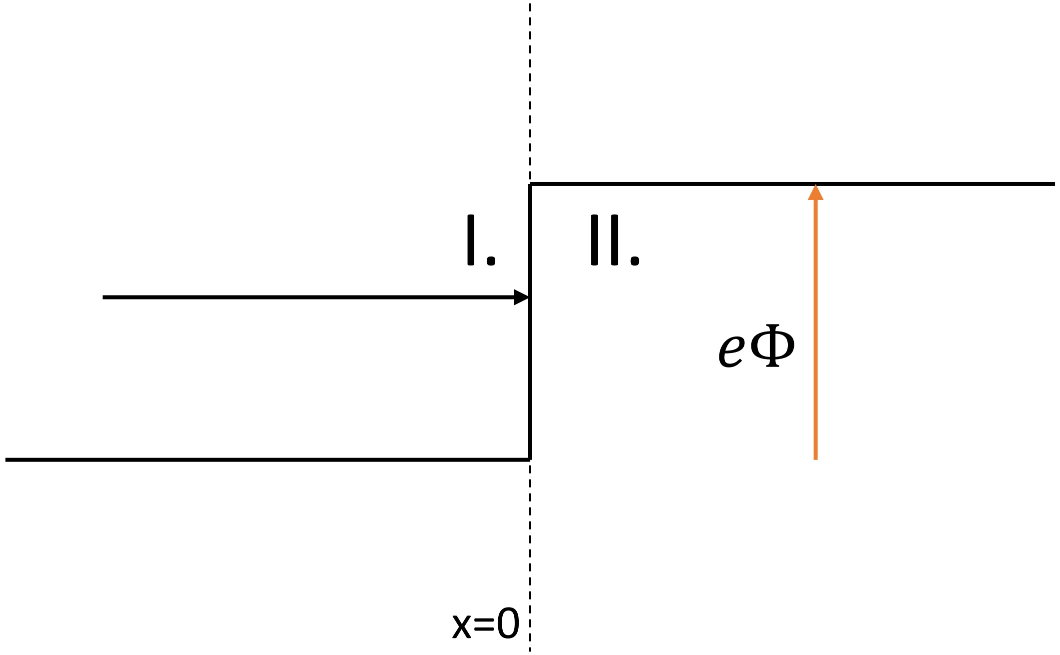
$$e\Phi(x) = V, x > 0$$

$$e\Phi(x) = 0, x < 0.$$

Calculate the transmission and reflection coefficients for the cases where $E < V < 2m$ and $V > 2m$ and interpret the results.


EMERGENCY EXIT
(chuckles)
I'm in danger.

The Solution (cont)



First, starting with the Dirac Equation:

$$(\vec{\alpha} \cdot \vec{p} + \beta m + e\Phi)\Psi(\vec{r}, t) = 0$$

We are looking at an electron, so we'll start by assuming that we are using the positive energy, right helicity solution for an electron.

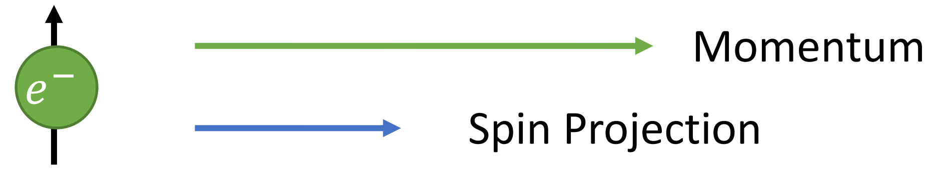
This general solution looks like:

$$\Psi_R^+ = e^{i(kx - Et)} \begin{pmatrix} 1 \\ 0 \\ \dfrac{k}{m + E} \\ 0 \end{pmatrix}$$

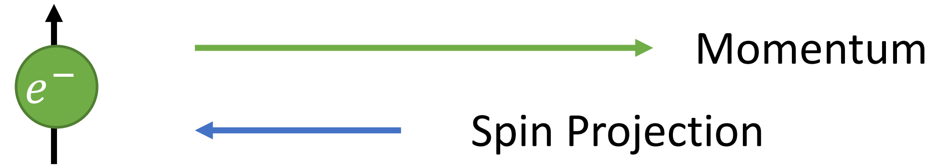
The Solution (cont)

How to define helicity:

Right-handed has spin projection parallel to momentum.

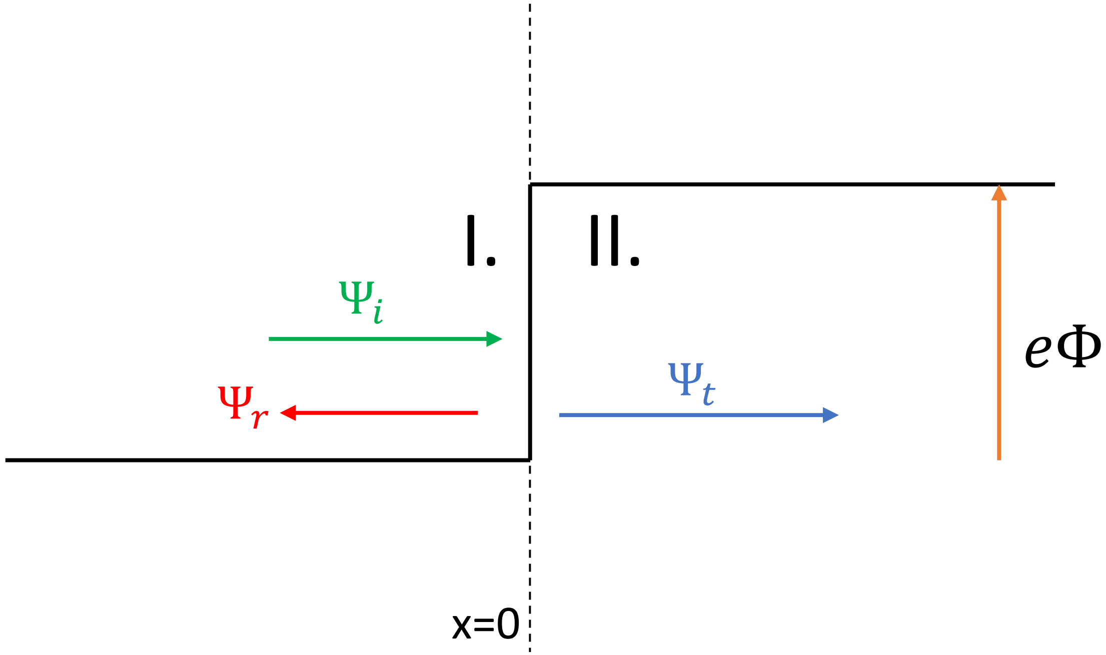


Left-handed has spin projection anti-parallel to momentum.



The Solution (cont)

We next need to write the solutions of the Dirac equation in both regions. Writing the incident, reflected, and transmitted wavefunctions separately.



$$\Psi_i(x) = e^{ikx}\begin{pmatrix} 1 \\ 0 \\ \frac{k}{E+m} \\ 0 \end{pmatrix}$$

$$\Psi_r(x) = be^{-ikx}\begin{pmatrix} 1 \\ 0 \\ \frac{-k}{E+m} \\ 0 \end{pmatrix} + b'e^{-ikx}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{k}{E+m} \end{pmatrix}$$

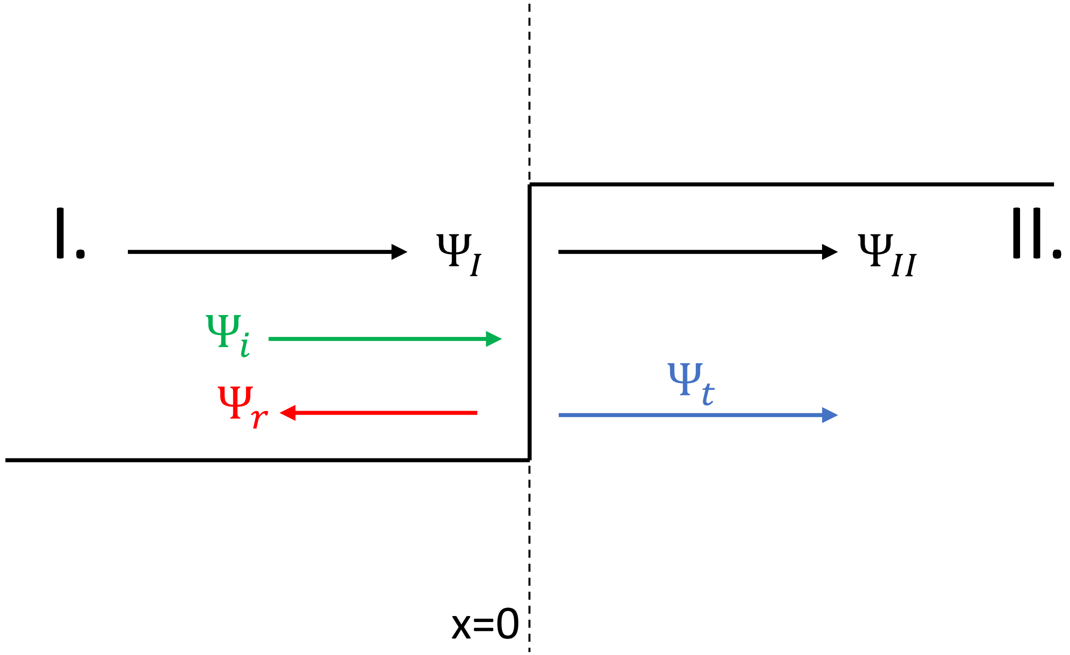
$$\Psi_t(x) = de^{iqx}\begin{pmatrix} 1 \\ 0 \\ \frac{q}{E-V+m} \\ 0 \end{pmatrix} + d'e^{-iqx}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{k}{E-V+m} \end{pmatrix}$$

Where $\mathrm{k} = \sqrt{E^2 - m^2}$ and

$$q = \sqrt{(E-V)^2 - m^2}$$

Note that $b' = d' = 0$ because we are assuming that the barrier isn't causing a spin-flip.

The Solution (cont)



After removing states (due to no spin-flip at the barrier).

$$k = \sqrt{E^2 - m^2}$$

$$q = \sqrt{(E-V)^2 - m^2}$$

Putting these together gives the following

$$\Psi_I(x) = \Psi_i + \Psi_r = e^{ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{k}{E+m} \\ 0 \end{pmatrix} + be^{-ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{-k}{E+m} \\ 0 \end{pmatrix}$$

$$\Psi_{II}(x) = de^{iqx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{q}{E-V+m} \\ 0 \end{pmatrix}$$

The Solution (cont)

Now, at the boundary, $\Psi_I(0) = \Psi_2(0)$

We match the respective spinor indices with each other. This will give two equations with 2 unknowns,

$$1 + b = d \qquad\qquad (1-b)\frac{k}{E+m} = d\ \frac{q}{E-V+m}$$

Now, divide the equations and solving for b and d (after some algebra/Mathematica simplification, whichever is preferred)

$$b = \frac{1-\xi}{1+\xi} \quad \text{and} \quad d = \frac{2}{1+\xi} \quad \text{where} \quad \xi = \frac{q}{k}\frac{(E+m)}{E-V+m}$$

The Solution (cont)

We want the transmission and reflection coefficients, which are defined in terms of currents as

$$R = -\frac{j_r}{j_i} \quad \text{and} \quad T = \frac{j_t}{j_i}$$

Using the definition of current from Sakurai to be

In 4-vector notation:

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi$$

where

$$\bar{\Psi} = \Psi^\dagger\beta$$

For the 3-vector component

$$\vec{j} = \Psi^\dagger\vec{\alpha}\Psi$$

where

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

We are working in 1-D, so we are going to use α_z, or

$$\alpha = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

The Solution (cont)

Solving for the incident, reflected, and transmitted current densities:

$$j_i = \Psi_{\mathrm{i}}^{\dagger}\alpha\Psi_{\mathrm{i}} = e^{-ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{k}{E+m} \\ 0 \end{pmatrix}^{T}\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} e^{ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{k}{E+m} \\ 0 \end{pmatrix} = 2\,\frac{k}{E+m}$$

$$j_r = \Psi_{\mathrm{r}}^{\dagger}\alpha\Psi_{\mathrm{r}} = be^{ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{-k}{E+m} \\ 0 \end{pmatrix}^{T}\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} be^{-ikx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{-k}{E+m} \\ 0 \end{pmatrix} = -2b^2\,\frac{k}{E+m}$$

$$j_t = \Psi_{\mathrm{t}}^{\dagger}\alpha\Psi_{\mathrm{t}} = de^{-iqx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{q}{E-V+m} \\ 0 \end{pmatrix}^{T}\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} de^{iqx}\begin{pmatrix} 1 \\ 0 \\ \dfrac{q}{E-V+m} \\ 0 \end{pmatrix} = 2d^2\,\frac{q}{E-V+m}$$

The Solution (cont)

Solving for the incident, reflected, and transmitted current densities:

$$j_i = 2\frac{k}{E+m} \qquad j_r = -2b^2\frac{k}{E+m} \qquad j_t = 2d^2\frac{q}{E-V+m}$$

For a sanity check, can also show that at the boundary

$$j_i + j_r = j_t$$

so the current is conserved.

$$R = -\frac{j_r}{j_i} = |b|^2 = \left|\frac{1-\xi}{1+\xi}\right|^2 \qquad T = \frac{j_t}{j_i} = |d|^2\frac{q}{k}\frac{E+m}{E-V+m}$$

Now, the different cases:

1) V<2m
2) V>2m

$$\xi = \frac{q}{k}\frac{(E+m)}{E-V+m}$$

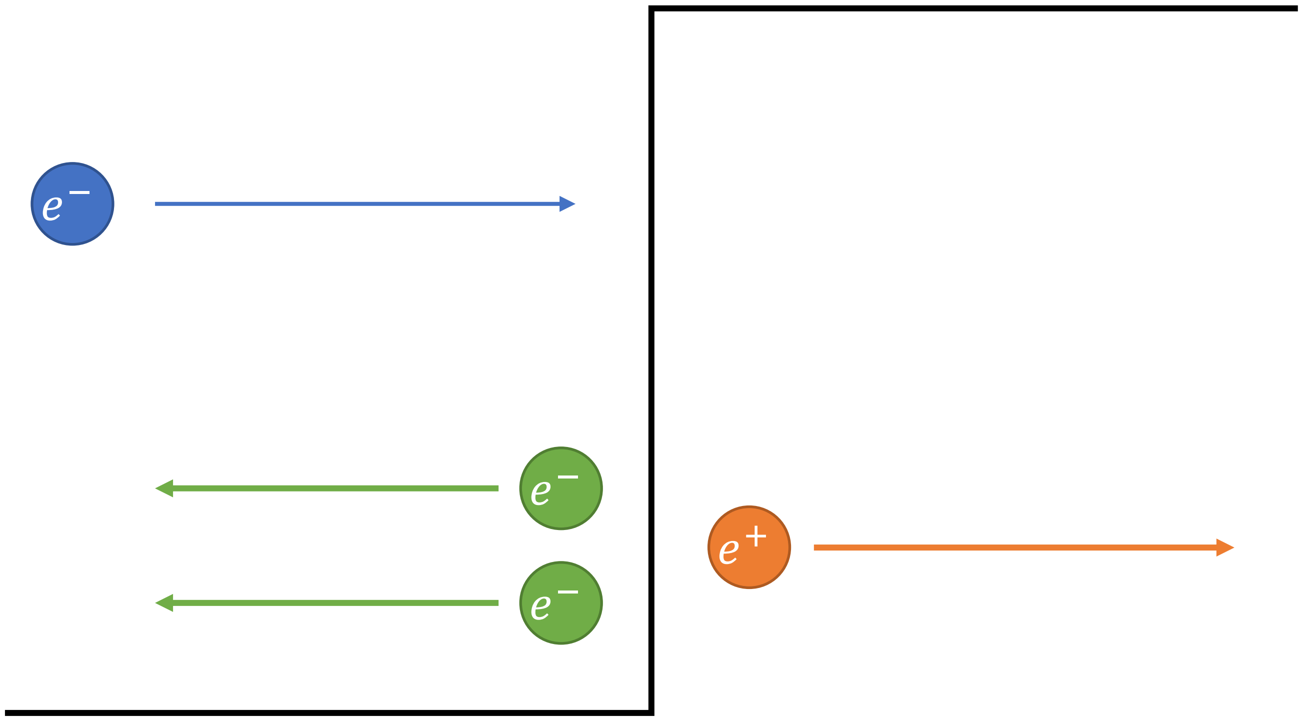
Now, the different cases:

1) For V<2m, everything is normal. $\xi > 0$, and like normal R+T = 1, and R,T will be between 0 and 1

2) For V>2m, weird stuff happens. $\xi < 0$, meaning that $b > 1$ which implies that R>1 and T<0. But how can this happen?
   →This is the Klein Paradox. Electron-holes are produced at the boundary

The Solution (cont)

2) For V>2m, weird stuff happens. $\xi < 0$, meaning that $b > 1$ which implies that R>1 and T<0. But how can this happen?
   →This is the Klein Paradox. Electron-holes are produced at the boundary



$$T = \frac{j_t}{j_i} = |d|^2 \frac{q}{k}\frac{E+m}{E-V+m}$$

$$R = -\frac{j_r}{j_i} = |b|^2 = \left|\frac{1-\xi}{1+\xi}\right|^2$$

$$\xi = \frac{q}{k}\frac{(E+m)}{E-V+m}$$