Chapter 8
Phase Shifts and Scattering

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Things you may need to know:

On (every) past equation sheet

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

Spherical Bessel (j),
Neumann (n) and Hankel (h) functions

*(Hankel functions h = j+/- i*n)

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2,$$

7.9

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k-\epsilon_r)^2+(\hbar\Gamma_R/2)^2},$$

7.49

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

7.21 7.19

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

8.5

σ - cross section
f - scattering amplitude

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

Legendre
Polynomials

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\frac{1}{k}P_\ell(\cos\theta)$$

8.15

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

8.15

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

8.18 8.19

Where we have seen it - Fall Final

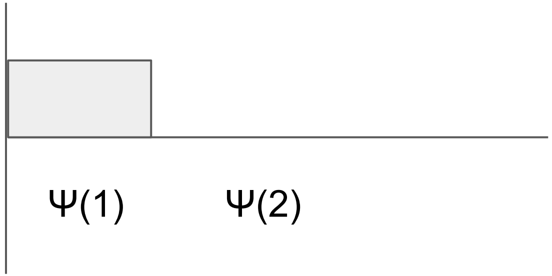
2. A particle of mass m scatters off a target with a spherically symmetric potential,

$$V(r) = V_0\Theta(R - r).$$

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the momentum p, where the energy is less than V_0.

(b) (5 pts) What is the cross-section in the limit that $p \to 0$?

Let’s solve this and also add on part c) the l=1 change in phase a)



$(hq)^2 = 2m(E - V0)$
$(hk)^2 = 2mE$

$\Psi(1) = \Psi(2)$ and $\Psi'(1) = \Psi'(2)$ at $r=R$

$A*\sinh(qx) = B \sin(kx + \delta)$ $\quad$ $q*A*\cosh(qx) = k*B*\cos(kx+\delta)$

Divide one equation by the other
(logarithmic derivative)

$1/q \tanh(qR) = 1/k \tan(kR + \delta)$

Some Algebra Later

$\delta = -kR + \tan^{-1}(k/q \tanh(qR))$

b) 8.19 $$\sigma = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \sin^2 \delta_\ell,$$

Part a) result:
$\delta = -kR + \tan^{-1}(k/q \tanh(qR))$

$\delta << 1$ $\sin^2(\delta) \to (\delta^2)$

$\sigma = 4\pi/(k^2)*(\delta^2) = 4\pi/(k^2)*(kR-k/q \tanh(qR))^2$

$k \to 0$ $\tan^{-1}(k/q \tanh(qR)) \to k/q \tanh(qR)$

$= 4\pi(R-1/q \tanh(qR))^2$

$= 4\pi R^2 - 8\pi R/q \tanh(qR) + 4\pi/q \tanh(qR)^2$

c)

$\Psi(1) = \Psi(2)$ and $\Psi'(1) = \Psi'(2)$ at r=R

Using the low energy limit (ch8 section 4) for the L=1 waves the radial waves become

$$R_\ell(k,r) \propto \left(h_\ell^*(kr) + e^{2i\delta} h_\ell(kr)\right)$$

and

$$j_\ell(kr) \to \frac{(kr)^\ell}{(2\ell+1)!!}$$

$$n_\ell(kr) \to \frac{(\ell-1)!!}{(kr)^{\ell+1}},$$

We get

$$\cot \delta_\ell(k) \approx (kb)^{-(2\ell+1)}(2\ell-1)!!(2\ell+1)!!\frac{\ell+1+b\alpha_\ell(k,b)}{\ell-b\alpha_\ell(k,b)} \qquad (8.43)$$

Where $\alpha = q \coth(qR)$

$\delta = \cot^{-1}[(kR)^{-3}(720)(2+Rq\ \mathrm{Coth}(qR))/(1- Rq\mathrm{Coth}(qR))]$

If the cross section is also wanted use ->

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

Scattering amplitude

$$f = \sum_{\ell=0}^{\infty} (2\ell + 1) f_\ell P_\ell(\cos\theta)$$

Using 1st order Born approximation, the scattering amplitude becomes:

Accurate if the scattered field is smaller than the incident one (weak scattering/high energies)

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = -\frac{m}{2\pi\hbar^2} \int d^3x' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} V(\mathbf{x}')$$

$$q = |\mathbf{k} - \mathbf{k}'| = 2k \sin\frac{\theta}{2}.$$

Where k’ is the outgoing scattered momentum and k is the incoming momentum

Through conservation of energy |k|=|k’|

Yukawa potential

$$V(r) = \frac{V_0 e^{-\mu r}}{\mu r}$$

This potential actually simplifies to the Coulomb potential when toying with the range μ

$$f^{(1)}(\theta) = -\frac{1}{2}\frac{2m}{\hbar^2}\frac{1}{iq}\int_0^\infty \frac{r^2}{r} V(r)(e^{iqr} - e^{-iqr})\,dr$$

$$= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin qr\,dr.$$

$$f^{(1)}(\theta) = -\left(\frac{2mV_0}{\mu\hbar^2}\right)\frac{1}{q^2+\mu^2},$$

Scattering amplitude

Differential cross section and wave function

Using this equation for the differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right) \simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)+\mu^2]^2}$$

Using the fact that:

$$q^2 = 4k^2\sin^2\frac{\theta}{2} = 2k^2(1-\cos\theta)$$

$$\psi(\vec{r}) \approx A\left[(e^{ipr}) + \frac{e^{ipr}}{r} f(\theta)\right]$$

We can approximate the outgoing scattered wavefunction as a sum of the incoming plane wave and small perturbation