PHY852 – Chapter 4: Harmonic Oscillator

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Consider a particle of mass $\boldsymbol{m}$ in a one-dimensional harmonic oscillator potential with fundamental frequency $\boldsymbol{\omega}$,

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

(a) Show that the Hamiltonian can be written as:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

(b) Using your answer from part (a), find the ground state energy of the harmonic oscillator.

(c) To second order in perturbation theory, find the correction to the ground state energy when the perturbation

$$V = \alpha X^2$$

is added to the system.

(a) Show: $H = \hbar\omega(a^+a + 1/2)$

Recalling: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$

consider a^+a:

$$a^+a = \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$= \frac{m\omega}{2\hbar}\left[x^2 + \underbrace{\frac{i}{m\omega}\hat{x}\hat{p} - \frac{i}{m\omega}\hat{p}\hat{x}}_{[\hat{x},\hat{p}] = i\hbar} + \frac{1}{m^2\omega^2}p^2\right]$$

$$= \frac{m\omega}{2\hbar}\left[x^2 + \frac{i}{m\omega}[\hat{x},\hat{p}] + \frac{p^2}{m^2\omega^2}\right]$$

$$= \frac{m\omega}{2\hbar}\left[x^2 - \frac{\hbar}{m\omega} + \frac{p^2}{m^2\omega^2}\right]$$

$$\Rightarrow a^+a = \frac{m\omega}{2\hbar}x^2 + \frac{p^2}{2\hbar m\omega} - \frac{1}{2} \longrightarrow \left(a^+a + \frac{1}{2}\right) = \frac{1}{\hbar\omega}\left(\underbrace{\frac{1}{2}m\omega^2x^2 + \frac{p^2}{2m}}_{= \hat{H}}\right)$$

$$\Rightarrow \boxed{\hat{H} = \hbar\omega(a^+a + 1/2)}$$

(b) To find the G.S. energy,

$$\hat{H}|n\rangle = E|n\rangle \qquad \left[\begin{array}{l} a^+ = \sqrt{n+1}\ |n+1\rangle \\ a = \sqrt{n}\ |n-1\rangle \end{array}\right]$$

$$\hat{H}|n\rangle = \hbar\omega(a^+a + 1/2)|n\rangle$$

$$= (\hbar\omega a^+a)|n\rangle + (\hbar\omega/2)|n\rangle$$

$$= (\hbar\omega)\sqrt{n}\, a^+|n-1\rangle + (\hbar\omega/2)|n\rangle$$

$$= (\hbar\omega)\sqrt{n}\sqrt{n}\,|n\rangle + (\hbar\omega/2)|n\rangle = \hbar\omega(n + 1/2)$$

$$\Rightarrow E_n = \hbar\omega(n + 1/2)$$

and for the ground state, $n=0$, $\boxed{E_0 = \hbar\omega/2}$

(c) $V = \alpha X^2$

(i) get X in terms of a & $a^\dagger$ and plug in for $V = \alpha X^2$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{X} + i\frac{1}{\sqrt{2\hbar m\omega}}\,\hat{P}$$

$$+\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{X} - i\frac{1}{\sqrt{2\hbar m\omega}}\,\hat{P}$$

$$(a + a^\dagger) = \sqrt{\frac{2m\omega}{\hbar}}\,\hat{X} \longrightarrow \hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$\Rightarrow X^2 = \frac{\hbar}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right)$$

$$\Rightarrow V = \alpha X^2 = \frac{\alpha\hbar}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right)$$

(ii) Find first order energy correction to G.S.

$$E_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle$$

$$E_0^{(1)} = \langle 0 | \alpha X^2 | 0 \rangle = \langle 0 | \frac{\alpha\hbar}{2m\omega}\left(\cancel{a^2} + \underline{\underline{aa^\dagger}} + \cancel{a^\dagger a} + \cancel{a^{\dagger 2}}\right) | 0 \rangle = \frac{\alpha\hbar}{2m\omega}\langle 0 | aa^\dagger | 0 \rangle$$

only surviving term

$$\Rightarrow \boxed{E_0^{(1)} = \frac{\alpha\hbar}{2m\omega}}$$

(iii) Find second order energy correction:

$$E_n^{(2)} = \sum_{m (\neq n)} \frac{|\langle m^{(0)} | V | n^{(0)} \rangle|^2}{E_n - E_m}$$

For some $\langle m | V | 0 \rangle$

$m \neq 0$, so $a^{\dagger 2}$ only surviving term and $\underline{m = 2}$.

$$\hookrightarrow = \frac{\alpha\hbar}{2m\omega}\langle m | \left(\cancel{a^2} + \cancel{aa^\dagger} + \cancel{a^\dagger a} + a^{\dagger 2}\right) | 0 \rangle \rightarrow \frac{\alpha\hbar}{2m\omega}\langle 2 | a^{\dagger 2} | 0 \rangle$$

$$\rightarrow \frac{\alpha\hbar}{2m\omega}\langle 2 | a^\dagger a^\dagger | 0 \rangle = \frac{\alpha\hbar}{2m\omega}\langle 2 | (\sqrt{1}) a^\dagger | 1 \rangle = \frac{\alpha\hbar}{2m\omega}\langle 2 | \sqrt{2} | 2 \rangle$$

$$= \frac{\alpha\hbar}{\sqrt{2}\, m\omega}$$

$$E_0^{(2)} = \frac{|\langle 2|V|0\rangle|^2}{\underset{\hbar\omega/2\uparrow}{E_0} - \underset{\nwarrow \text{ from}}{E_2}} = \left(\frac{\alpha^2\hbar^2}{2m^2\omega^2}\right)\left(\frac{1}{\hbar\omega/2 - 5\hbar\omega/2}\right) = \left(\frac{\alpha^2\hbar^2}{2m^2\omega^2}\right)\left(\frac{1}{-2\hbar\omega}\right)$$

$E_n = \hbar\omega(n+1/2)$

$E_2 = 5\hbar\omega/2$

$$\boxed{E_0^{(2)} = \frac{-\alpha^2\hbar}{4m^2\omega^3}}$$

$$\Rightarrow \boxed{E_0 = \frac{m\omega}{2} + \frac{\alpha\hbar}{2m\omega} - \frac{\alpha^2\hbar}{4m^2\omega^3}}$$