The formal statement of Wigner-Eckhart theorem is:

$$\langle \beta_f, l_f, m_f | T_q^k | \beta_i, l_i, m_i \rangle = C^{k,l_i}_{q,m_i;l_f,m_f} \frac{\langle \beta_f, l_f || T^{(k)} || \beta_i, l_i \rangle}{\sqrt{2l_f+1}}$$

Double bars for the reduced matrix element in the numerator are only to indicate that the m values don't affect anything. $k$ is the "rank" of the irreducible tensor operator $T$. Selection rules (can be derived from looking at Clebsch-Gordon table):

$$\Delta m = m_f - m_i = q$$
$$|l_i - k| \leq l_f \leq l_i + k$$

Naturally, the theorem has a great utility in finding the overlap between angular momentum states for given operators. In order to make use of the theorem, one must represent a given operator in terms of irreducible tensor operators. Tensor operators are "irreducible" because there is no smaller set of them that can span the vector space these operators exist in. Recall how in electrostatics we shifted from the nine-component quadrupole tensor to the five spherical harmonics with $l = 2$. The irreducible tensor operators work just like the spherical harmonics, in that they compose the minimum spanning set of operators to cover $R^3$. One can construct a tensor operator through the relation:

$$T_q^k = r^k Y_q^k \sqrt{4\pi(2k+1)}$$

Here $Y_q^k$ are the usual spherical harmonics $Y_m^l$ that we are used to, simply relabeled. As an explicit example, let's construct the irreducible representation for the operator $P_x$:

1. $P_x + iP_y = Psin(\theta)(cos(\phi) + isin(\phi)) = Psin(\theta)e^{i\phi}$
2. $P_x - iP_y = Psin(\theta)(cos(\phi) - isin(\phi)) = Psin(\theta)e^{-i\phi}$
3. observe that $Y_{\pm 1}^{1} = \mp\sqrt{\frac{3}{8\pi}} sin(\theta)e^{\pm i\phi}$ so if we take $r \longrightarrow P$:
   $T_{\pm 1}^{1} = \mp\sqrt{\frac{9}{2}} Psin(\theta)e^{\pm i\phi} \implies P_x \pm iP_y = \mp\frac{\sqrt{2}}{3} T_{\pm 1}^{1}$
4. $P_x = \frac{1}{2}[(P_x + iP_y) + (P_x - iP_y)] = \frac{\sqrt{2}}{6}(T_{-1}^{1} - T_1^1)$

Wigner-Eckart is good for any vector operator ($\vec{r}$, $\vec{p}$, $\vec{L}$, etc), since they can be represented with the irreducible tensor operators. Scalar operators, e.g. $r^2$, are a special case of the selection rules, as they can be represented as $r^2 T_0^0$ (since $T_0^0 = 1$). Scalar operators are defined as being invariant under rotation ($r^2, p^2, ...$) Therefore:

$$\langle \beta_f, l_f, m_f | r^2 | \beta_i, l_i, m_i \rangle = \langle \beta_f, l_f, m_f | r^2 T_0^0 | \beta_i, l_i, m_i \rangle \propto \delta_{m_i,m_f} \delta_{l_i,l_f}$$

Subject exam questions have always been focused on selection rules, identifying which matrix elements are zero or nonzero, but we could also see questions like those we did in our quizzes: given one matrix element, find some others in terms of Clebsch-Gordon coefficients.
Solving for a different matrix element:

1. Solve for the garbage in terms of the given matrix element
   $$M_0 = \langle \beta_f, l_f, m_f | T_q^k | \beta_i, l_i, m_i \rangle = C^{k,l_i}_{q,m_i;l_f,m_f} \underbrace{\frac{\langle \beta_f, l_f || T^{(k)} || \beta_i, l_i \rangle}{\sqrt{2l_f+1}}}_{\xi}$$
   $$\implies \xi = \frac{M_0}{C^{k,l_i}_{q,m_i;l_f,m_f}}$$
2. $\xi$ will be the same for all operators between states with the same $l_i$ and $l_f$, so you can solve for other operators in terms of $\xi$
   $$\langle \beta_f, l_f, m_f | T_b^a | \beta_i, l_i, m_i \rangle = C^{a,l_i}_{b,m_i;l_f,m_f} \xi = \frac{C^{k,l_i}_{q,m_i;l_f,m_f}}{C^{a,l_i}_{b,m_i;l_f,m_f}} M_0$$

Be wary of trying to use Wigner-Eckhart to find the overlap between states for different observables. Even though $x$ and $P_x$ share the same irreducible representation, the reduced matrix element associated with these two operators will be different in general.

Example problem:

1. Identify which of the following matrix elements are nonzero:
   (a) $\langle \beta_f, l_f = 0, m_f | P^2 | \beta_i, l_i = 1, m_i \rangle$
   (b) $\langle \beta_f, l_f = 1, m_f | L_z | \beta_i, l_i = 3, m_i \rangle$
   (c) $\langle \beta_f, l_f = 1, m_f | x^2 + y^2 | \beta_i, l_i = 4, m_i \rangle$
   (d) $\langle \beta_f, l_f = 0, m_f | P_y^2 + P_z^2 | \beta_i, l_i = 3, m_i \rangle$
   (e) $\langle \beta_f, l_f = 0, m_f | x | \beta_i, l_i = 1, m_i \rangle$
2. Using the Wigner-Eckart theorem, evaluate all the nonzero matrix elements above for all possible $m$ combinations for a particle in an infinite spherical well of radius R. Assume the particle is in the ground state for the given angular momentum. (Hint: You can write $\psi(\vec{r}) = \frac{u(r)}{r} Y_{lm}(\theta, \phi)$, where $u(r)$ is found using the radial Schrödinger equation)

Solution:

1. Using the $l$ selection rules ($|l_i - k| \leq l_f \leq l_i + k$) and our knowledge of the ranks of the tensor operators these matrix elements comprise:
   (a) $P^2$ is rotationally invariant, making it a scalar operator that we can rewrite as $P^2 T_0^0$. $T_0^0$ can only connect states with identical $l$ and $m$ values, so this is 0
   (b) $L_z$ goes as $T_0^1$, meaning it can only connect $l$ values that differ by one. This is 0.
   (c) $x^2$ and $y^2$ go as rank 2 tensors, so they can only connect $l$ states that differ by 2 or less. This is 0.
   (d) For the same reason as (c), this is 0.
   (e) $x$ is a linear combination of rank one tensors, meaning it can connect $l$ states that differ by 1. This is nonzero.
2. The Wigner-Eckart theorem shows that, once one matrix element is known for a given combination of $l_i$, $l_f$, any other matrix element for that same combination can be found with ratios of Clebsch-Gordon coefficients. This means that we can choose the simplest case to solve the integral, and then convert that to the matrix elements we want.
   (a) Find the wavefunction. The radial wavefunction in this case is given by the spherical Bessel functions $j_l(r)$ times the spherical harmonics and a normalization factor. The process of solving for $u(r)$ is outlined in section 4.10, and is quite similar to that of solving for a normal 1D wavefunction. We find, with generous application of Mathematica, that
   $\psi_i(\vec{r}) = \frac{6.5101}{R^{3/2}} j_1(\frac{z_1}{R} r) Y_{lm}(\theta, \phi), \quad \psi_f(\vec{r}) = \frac{\sqrt{2\pi}}{R^{3/2}} j_0(\frac{z_0}{R} r) Y_{lm}(\theta, \phi)$
   where $z_1$ and $z_0$ denote the first root of $j_1$ and $j_0$ respectively.
   (b) Find the easiest matrix element. We'll choose to find
   $\langle \beta_f, l_f = 0, m_f = 0 | z | \beta_i, l_i = 1, m_i = 0 \rangle$
   because this integral is $\phi$ independent, and then use Wigner-Eckart to convert to the x matrix elements. We can then find the $z$ matrix element by taking the integral:
   $\int_0^R dr \int_0^\pi d\theta \, 2\pi \psi_i(\vec{r})(r\cos\theta)\psi_f(\vec{r}) r^2 \sin\theta \approx 0.306R \equiv M_0$
   (c) Use Wigner-Eckart to get the matrix elements we want. We now have $M_0 = \langle \beta_f, l_f = 0, m_f = 0 | z | \beta_i, l_i = 1, m_i = 0 \rangle = C^{1,1}_{0,0;0,0} \xi$
   where $\xi$ stands for the reduced matrix element times its normalization (Note that $z = T_0^1$). We then get $\xi = \frac{M_0}{C^{1,1}_{0,0;0,0}}$.
   We know that
   $x = \frac{1}{\sqrt{2}}(-T_1^1 + T_{-1}^1)$, so
   $$\langle \beta_f, l_f = 0, m_f | x | \beta_i, l_i = 1, m_i \rangle =$$
   $$= \frac{1}{\sqrt{2}} \underbrace{\langle \beta_f, l_f = 0, m_f | T_{-1}^1 | \beta_i, l_i = 1, m_i \rangle}_{C^{1,1}_{-1,m_i;0,m_f}\xi} - \frac{1}{\sqrt{2}} \underbrace{\langle \beta_f, l_f = 0, m_f | T_1^1 | \beta_i, l_i = 1, m_i \rangle}_{C^{1,1}_{1,m_i;0,m_f}\xi}$$
   $$= \frac{1}{\sqrt{2}} \left( \frac{C^{1,1}_{1,m_i;0,m_f} - C^{1,1}_{1,m_i;0,m_f}}{C^{1,1}_{0,0;0,0}} \right) M_0$$
   $$\approx 0.216R \left( \frac{C^{1,1}_{1,m_i;0,m_f} - C^{1,1}_{1,m_i;0,m_f}}{C^{1,1}_{0,0;0,0}} \right)$$
   You can then plug in values for $m_i$ and $m_f$ as necessary to get all the different matrix elements.