An electron is in an $\ell = 1$ state of a hydrogen atom. It experience a spin orbit interaction

$$V_{so} = \alpha \vec{L} \cdot \vec{S}$$

and feels a uniform external magnetic field

$$V_b = \mu \vec{B} \cdot \left(\vec{L} + 2\vec{S}\right)$$

Calculate the eigenvalues of the Hamiltonian in the $|JM\rangle$ basis.

Calculate the Clebsch-Gordan coefficients.

$$J_-|J, M\rangle = \hbar\sqrt{(J+M)(J-M+1)}|J, M-1\rangle$$
$$S_-|s, m_s\rangle = \hbar\sqrt{(s+m_s)(s-m_s+1)}|s, m_s-1\rangle$$
$$L_-|\ell, m_\ell\rangle = \hbar\sqrt{(\ell+m_\ell)(\ell-m_\ell+1)}|\ell, m_\ell-1\rangle$$

$$\boxed{\left|J = \frac{3}{2}, M = \frac{3}{2}\right\rangle = \left|m_\ell = 1, m_s = \frac{1}{2}\right\rangle}$$

$$J_-\left|J = \frac{3}{2}, M = \frac{3}{2}\right\rangle = (S_- + L_-)\left|m_\ell = 1, m_s = \frac{1}{2}\right\rangle$$
$$\hbar\sqrt{3}\left|J = \frac{3}{2}, M = \frac{1}{2}\right\rangle = \hbar\left|m_\ell = 1, m_s = -\frac{1}{2}\right\rangle + \hbar\sqrt{2}\left|m_\ell = 0, m_s = \frac{1}{2}\right\rangle$$

$$\boxed{\left|J = \frac{3}{2}, M = \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|m_\ell = 1, m_s = -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|m_\ell = 0, m_s = \frac{1}{2}\right\rangle}$$

by orthogonality (switch the coefficients and make one negative):

$$\boxed{\left|J = \frac{1}{2}, M = \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_\ell = 1, m_s = -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|m_\ell = 0, m_s = \frac{1}{2}\right\rangle}$$

etc...

$$\boxed{\left|J = \frac{3}{2}, M = -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|m_\ell = -1, m_s = \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|m_\ell = 0, m_s = -\frac{1}{2}\right\rangle}$$

$$\boxed{\left|J = \frac{1}{2}, M = -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_\ell = -1, m_s = \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|m_\ell = 0, m_s = -\frac{1}{2}\right\rangle}$$

$$\boxed{\left|J = \frac{3}{2}, M = -\frac{3}{2}\right\rangle = \left|m_\ell = -1, m_s = -\frac{1}{2}\right\rangle}$$

We can use these relations to form a matrix that transforms a $|m_\ell, m_s\rangle$ state into a $|JM\rangle$ state.

$$\begin{pmatrix} \left|\frac{3}{2},\frac{3}{2}\right\rangle \\ \left|\frac{3}{2},\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},\frac{1}{2}\right\rangle \\ \left|\frac{3}{2},-\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \\ \left|\frac{3}{2},-\frac{3}{2}\right\rangle \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\ & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\ & & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & \\ & & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} \left|1,\frac{1}{2}\right\rangle \\ \left|1,-\frac{1}{2}\right\rangle \\ \left|0,\frac{1}{2}\right\rangle \\ \left|-1,\frac{1}{2}\right\rangle \\ \left|0,-\frac{1}{2}\right\rangle \\ \left|-1,-\frac{1}{2}\right\rangle \end{pmatrix}$$

$$\left|\psi^{(J)}\right\rangle = C\left|\psi^{(\ell s)}\right\rangle$$

Any permutation of this matrix also works, as long as the basis vectors are adjusted accordingly. Since the Clebsch-Gordan coefficients are real (by convention), the transpose of this matrix is its inverse.

$$C^T = C^{-1}$$

This can be demonstrated by solving for the $|m_\ell, m_s\rangle$ states in terms of the $|JM\rangle$ states. In this case,

$$C = C^T = C^{-1}$$

but this is not true in general.

Now we write down the Hamiltonian. The spin-orbit component is diagonal in the $|JM\rangle$ basis.

$$\vec{J}^2 = \left(\vec{L} + \vec{S}\right)^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \Rightarrow \frac{\hbar^2}{2}\left(J\left(J+1\right) - \ell\left(\ell+1\right) - s\left(s+1\right)\right)$$

$$\left\langle\frac{3}{2},\frac{3}{2}\middle|V_{so}\middle|\frac{3}{2},\frac{3}{2}\right\rangle = \left\langle\frac{3}{2},\frac{1}{2}\middle|V_{so}\middle|\frac{3}{2},\frac{1}{2}\right\rangle = \left\langle\frac{3}{2},-\frac{1}{2}\middle|V_{so}\middle|\frac{3}{2},-\frac{1}{2}\right\rangle = \left\langle\frac{3}{2},-\frac{3}{2}\middle|V_{so}\middle|\frac{3}{2},-\frac{3}{2}\right\rangle = \frac{\alpha\hbar^2}{2}$$

$$\left\langle\frac{1}{2},\frac{1}{2}\middle|V_{so}\middle|\frac{1}{2},\frac{1}{2}\right\rangle = \left\langle\frac{1}{2},-\frac{1}{2}\middle|V_{so}\middle|\frac{1}{2},-\frac{1}{2}\right\rangle = -\alpha\hbar^2$$

All off-diagonal matrix elements give zero.

$$V_{so}^{(J)} = \frac{\alpha\hbar^2}{2}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -2 & & & \\ & & & 1 & & \\ & & & & -2 & \\ & & & & & 1 \end{pmatrix}$$

In contrast, the component due to the magnetic field is diagonal in the $|m_\ell, m_s\rangle$ basis.

$$\left\langle 1,\tfrac{1}{2}\middle|V_b\middle|1,\tfrac{1}{2}\right\rangle = 2\mu\hbar B \qquad \left\langle 1,-\tfrac{1}{2}\middle|V_b\middle|1,-\tfrac{1}{2}\right\rangle = 0 \qquad \left\langle 0,\tfrac{1}{2}\middle|V_b\middle|0,\tfrac{1}{2}\right\rangle = \mu\hbar B$$

$$\left\langle -1,\tfrac{1}{2}\middle|V_b\middle|-1,\tfrac{1}{2}\right\rangle = 0 \qquad \left\langle 0,-\tfrac{1}{2}\middle|V_b\middle|0,-\tfrac{1}{2}\right\rangle = -\mu B \qquad \left\langle -1,-\tfrac{1}{2}\middle|V_b\middle|-1,-\tfrac{1}{2}\right\rangle = -2\mu\hbar B$$

$$V_b^{(\ell s)} = \mu\hbar B \begin{pmatrix} 2 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & -1 & \\ & & & & & -2 \end{pmatrix}$$

We perform a transformation to the $|JM\rangle$ basis using the matrix of Clebsch-Gordan coefficients.

$$V_b^{(J)} = CV_b^{(\ell s)}C^T = \mu\hbar B \begin{pmatrix} 2 & & & & & \\ & \frac{2}{3} & -\frac{\sqrt{2}}{3} & & & \\ & -\frac{\sqrt{2}}{3} & \frac{1}{3} & & & \\ & & & -\frac{2}{3} & \frac{\sqrt{2}}{3} & \\ & & & \frac{\sqrt{2}}{3} & -\frac{1}{3} & \\ & & & & & -2 \end{pmatrix}$$

Behold:

$$H^{(J)} = V_{so}^{(J)} + V_b^{(J)}$$

$$= \begin{pmatrix} \frac{\alpha\hbar^2}{2} + 2\mu\hbar B & & & & & \\ & \frac{\alpha\hbar^2}{2} + \frac{2}{3}\mu\hbar B & -\frac{\sqrt{2}}{3}\mu\hbar B & & & \\ & -\frac{\sqrt{2}}{3}\mu\hbar B & -\alpha\hbar^2 + \frac{1}{3}\mu\hbar B & & & \\ & & & \frac{\alpha\hbar^2}{2} - \frac{2}{3}\mu\hbar B & \frac{\sqrt{2}}{3}\mu\hbar B & \\ & & & \frac{\sqrt{2}}{3}\mu\hbar B & -\alpha\hbar^2 - \frac{1}{3}\mu\hbar B & \\ & & & & & \frac{\alpha\hbar^2}{2} - 2\mu\hbar B \end{pmatrix}$$

By choosing unit vectors such that the matrix of Clebsch-Gordan coefficients is block-diagonal, the eigenvalues are easily calculated:

$$E = \begin{cases} \frac{\alpha\hbar^2}{2} \pm 2\mu\hbar B \\ \frac{\mu\hbar B}{2} - \frac{\alpha\hbar^2}{4} \pm \sqrt{\left(\frac{\alpha\hbar^2}{4} + \frac{\mu\hbar B}{2}\right)^2 + \frac{\alpha^2\hbar^4}{2}} \\ -\frac{\mu\hbar B}{2} - \frac{\alpha\hbar^2}{4} \pm \sqrt{\left(\frac{\alpha\hbar^2}{4} - \frac{\mu\hbar B}{2}\right)^2 + \frac{\alpha^2\hbar^4}{2}} \end{cases}$$