your name(s)____

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Physics 852 Quiz #18 - Friday, April. 3rd

Chirality

Consider the chirality operator,

$$\gamma_5 = i\gamma_0\gamma_x\gamma_y\gamma_z = i\beta\beta\alpha_x\beta\alpha_y\beta\alpha_z = -i\alpha_x\alpha_y\alpha_z. \tag{0.1}$$

1. Show that γ_5 is Hermitian.
2. Show that $\gamma_5^2 = \mathbb{I}$.
3. What are the eigenvalues of γ_5
4. Show that $(1+\gamma_5)/2$ and $(1-\gamma_5)/2$ are projection operators.
5. Show that γ_5 commutes with the Hamiltonian for massless particles,
$$H = \vec{\alpha}\cdot\vec{p},$$
but does not commute with H if a mass term
$$H_M = \beta m$$
is added.
6. Write γ_5 in the chiral representation.
7. "Prove" that
$$\frac{1}{3!}\sum_{ijk}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k\alpha_\ell = \frac{1}{2}\sum_{ij}\epsilon_{ij\ell}\alpha_i\alpha_j.$$
You can use the fact that $\sum_{ijk}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k$ is rotationally invariant.
8. For massless particles, the Dirac equation is
$$(\vec{\alpha}\cdot\hat{p})u_{\vec{p},s} = u_{\vec{p},s}$$
$$(\vec{\alpha}\cdot\hat{p})v_{-\vec{p},s} = -v_{-\vec{p},s}.$$
Exploiting the information above, show that for massless particles,
$$\gamma_5 u_{\vec{p},s} = \frac{1}{2}(\vec{\Sigma}\cdot\hat{p})u_{\vec{p},s},$$
$$\gamma_5 v_{-\vec{p},s} = -\frac{1}{2}(\vec{\Sigma}\cdot\hat{p})v_{-\vec{p},s}.$$

Comment: In the standard model the weak interaction couples only to neutrinos of a given chirality, e.g. the terms coupling to neutrinos appears as $(1-\gamma_5)\gamma_\mu\Psi(x)$. The operator γ_5 has odd parity, so the operator $(1-\gamma_5)$ mixes even and odd parity maximally. Thus, in the famous experiment of Chien-Shiung Wu `https://en.wikipedia.org/wiki/Chien-Shiung_Wu`, the direction of neutrinos (a vector) lined up with the direction of the magnetic field (a pseudo vector) thus demonstrating that in the weak interaction the choice of right-handed vs. left-handed coordinate systems is no longer arbitrary, and represents a striking violation of parity conservation.

① $\gamma_5 = -i\,\alpha_x \alpha_y \alpha_z\,,\quad \gamma_5^+ = i\,\alpha_z^+ \alpha_y^+ \alpha_x^+ = i\,\alpha_z \alpha_y \alpha_x$

$$= i\,\alpha_y \alpha_x \alpha_z = -i\,\alpha_x \alpha_y \alpha_z$$

$$= \gamma_5 \checkmark$$

② $\gamma_5^2 = -\alpha_x \alpha_y \alpha_z \alpha_x \alpha_y \alpha_z$

2 place

$$= -\alpha_y \alpha_z \alpha_y \alpha_z$$

1 place

$$= +1$$

③ eigenvalues must be ± 1 because $\gamma_5^2 = 1$

④ $\left(\left(\frac{1+\gamma_5}{2}\right)\right)^2 = \frac{1}{4} + \frac{\gamma_5^2}{4} + \frac{2\gamma_5}{4} = \left(\frac{1+\gamma_5}{2}\right) \checkmark$

$P^2 = P$

⑤ $[\gamma_5, \alpha_x] = -i\,\alpha_x \alpha_y \alpha_z \alpha_x$ 2 place

$$+ i\,\alpha_x^2 \alpha_y \alpha_z$$

$$= -i\,\alpha_y \alpha_z + i\,\alpha_y \alpha_z = 0$$

must be true for α_y, α_z also

⑥ $\gamma_5 = -i \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix} \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$

$$= -i \cdot i \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

7. "Prove" that
$$\frac{1}{3!}\sum_{ijk}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k\alpha_\ell = \frac{1}{2}\sum_{ij}\epsilon_{ij\ell}\alpha_i\alpha_j.$$
You can use the fact that $\sum_{ijk}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k$ is rotationally invariant.

(7.)

Choose $\ell = x$, so must prove

$$\frac{1}{3!}\left(\alpha_x\alpha_y\alpha_z + \alpha_y\alpha_z\alpha_x + \alpha_z\alpha_x\alpha_y - \alpha_y\alpha_x\alpha_z - \alpha_x\alpha_z\alpha_y - \alpha_z\alpha_y\alpha_x\right)\alpha_x$$

all terms the same

$$\stackrel{?}{=} \frac{1}{2}\left(\alpha_y\alpha_z - \alpha_z\alpha_y\right)$$

same

$$\alpha_x\alpha_y\alpha_z\alpha_x \stackrel{?}{=} \alpha_y\alpha_z$$

2 places

$$\alpha_y\alpha_z \stackrel{?}{=} \alpha_y\alpha_z \quad \checkmark$$

if true for $\ell = x$ must be true for all ℓ by rotational invariance

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(8.) From (7) $\frac{1}{2}\gamma_5\alpha_\ell = \Sigma_\ell$

$$\frac{1}{2}\gamma_5\vec{\alpha}\cdot\hat{p} = \vec{\Sigma}\cdot\hat{p}$$

$$\frac{1}{2}\gamma_5\vec{\alpha}\cdot\hat{p}\,u_s(\vec{p}) = (\vec{\Sigma}\cdot\hat{p})\,u_s(\vec{p})$$

$$\frac{1}{2}\gamma_5 u_s(\vec{p}) = (\vec{\Sigma}\cdot\hat{p})\,u_s(\vec{p})$$

$$\frac{1}{2}\gamma_5\vec{\alpha}\cdot\hat{p}\,v_s(-\vec{p}) = \vec{\Sigma}\,\hat{p}\,v_s(-\vec{p})$$

$$-\frac{1}{2}\gamma_5 v_s(-\vec{p}) = \vec{\Sigma}\cdot\hat{p}\,v_s(-\vec{p})$$

OR

$$\frac{1}{2}\gamma_5 v_s(+\vec{p}) = \vec{\Sigma}\cdot\hat{p}\,v_s(+\vec{p})$$