Chapter 1

1. Suppose you are doing a fixed-target experiment at the LHC. The protons have a beam energy of 7 TeV. (The proton mass is 938.28 MeV/c^2). If the experiment were redone with a collider built with an equivalent center-of-mass energy, what would that energy be.

The problem states that we perform a fixed target experiment where the incident beam has an energy of 7 TeV. Then it asks what the beam energy would be if we boost to the CM frame. In what follows, we will refer to the fixed target frame as the lab frame.

In the Lab frame we define two four vectors: one for the beam and one for the fixed target, each of mass m.

$$P_1^\mu = (E, p, 0, 0) \tag{0.1}$$

$$P_2^\mu = (m, 0, 0, 0) \tag{0.2}$$

Next, we calculate the Lorentz Invariant quantity, s, the square of the center of mass energy.

$$s = (P_1^\mu + P_2^\mu)^2 \tag{0.3}$$

$$s = (P_1^\mu)^2 + 2P_1^\mu P_{2\ \mu} + (P_2^\mu)^2 \tag{0.4}$$

$$s = m^2 + 2(Em) + m^2 \tag{0.5}$$

where we have used the fact that $E^2 - |\vec{p}|^2 = m^2$ in the previous step; therefore, s is given by

$$s = 2m(E + m) \tag{0.6}$$

With s known in the lab frame, we now want to calculate it in the CM frame. Again we define two four vectors: one for each beam, mass m.

$$P_1'^\mu = (E', p', 0, 0) \tag{0.7}$$

$$P_2'^\mu = (E', -p', 0, 0) \tag{0.8}$$

In the CM frame, recall the momenta are equal and opposite and the energy of each beam is the same, then, again forming s,

$$s = (P_1'^\mu + P_2'^\mu)^2 \tag{0.9}$$

$$s = 4E'^2 \tag{0.10}$$

Since s is invariant, we can equate its expression in the lab frame to the CM frame, in order to solve for the beam energy in the CM frame.

$$2m(E+m) = 4E'^{2} \tag{0.11}$$

Finally, with a little algebra we get E' in terms of E

$$E' = \sqrt{\frac{m(m+E)}{2}} \tag{0.12}$$

After plugging in the numbers we find $E' = 0.057$ TeV.

2. This is very similar to problem 1. We start in the CM frame and want to boost to the fixed target frame. In problem 1 we already defined the relevant four vectors and calculated the invariant, s, in each frame. Here we want to solve for the energy in the lab frame in terms of the CM energy.

$$s = 2m(E+m) = 4E'^2 \tag{0.13}$$

Rearranging and solving for E

$$E = \frac{4E'^2 - 2m^2}{2m} = \frac{2E'^2 - m^2}{m} \tag{0.14}$$

Plugging in the relevant values results in $E = 1.04 \cdot 10^5$ TeV.

3. Consider a $1+1$ dimension vector, (E, p), where $m^2 = E^2 - p^2$. Consider the transformed vector, $p'_\alpha = \mathcal{L}_{\alpha\beta}\, p^\beta$, show that $m'^2 = E'^2 - p'^2 = m^2$.
We are given the vector, (E, p), where $m^2 = E^2 - p^2$, and are told to show that $m^{'2} = E^{'2} - p^{'2} = m^2$.

First, we will use the given definition of $p^{'}_{\alpha}$ to construct that four vector.

$$p'_\alpha = \text{Ł}_{\alpha\beta} p^\beta \tag{0.15}$$

$$p'_\alpha = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix} = \begin{pmatrix} E\cosh\eta + p\sinh\eta \\ E\sinh\eta + p\cosh\eta \end{pmatrix} \tag{0.16}$$

Since $m'^2 = E'^2 - p'^2 = p'_\alpha p'^\alpha$,

$$p'_\alpha p'^\alpha = (E\cosh\eta + p\sinh\eta)^2 - (E\sinh\eta + p\cosh\eta)^2 \tag{0.17}$$

After some algebra and utilizing the fact that $\cosh\eta^2 - \sinh\eta^2 = 1$, you are left with

$$E^2 + p^2 = m^2 \tag{0.18}$$

4. Consider two particle with four-momenta p_a and p_b. Particle a is recorded at the space time point $r_a = (0, 0, 0, 0)$ and particle b is recorded at $r_b = r$. For an observer moving with particle a find the time at which particle b passes at the point of closest approach. Express your answer in terms of Lorentz invariants, i.e., dot products involving p_a , p_b and r.

So, for this problem, we have two particles with four momenta P_a and P_b and positions r_a = (0,0,0,0) and r_b = r. We are moving with particle a, so we need to set up projectors in this frame.

$$r'^{\alpha} = r^{\alpha} - \frac{P_a^{\alpha}(P_a \cdot r)}{P_a^2} \tag{0.19}$$

$$P_b'^{\alpha} = P_b^{\alpha} - \frac{P_a^{\alpha}(P_a \cdot P_b)}{P_a^2} \tag{0.20}$$

These projectors remove the $\alpha = 0$ component of the position and momentum in the center of mass frame because they are written in terms of invariants. The energy is defined by:

$$E_b = \frac{P_a \cdot P_b}{\sqrt{P_a^2}}, \tag{0.21}$$

since the energy is the first component of the momentum four vector. In terms of these invariant quantities, the time of closest approach is defined as:

$$t = \frac{E_b(r' \cdot P_b')}{P_b'^2}. \tag{0.22}$$

After plugging in the appropriate values and performing some simplifying algebra, the time of closest approach comes out to be

$$t = -\frac{(P_a \cdot P_b)((r \cdot P_b) - \frac{(r \cdot P_a)(P_a \cdot P_b)}{P_a^2})}{\sqrt{P_a^2}(P_b^2 - \frac{(P_a \cdot P_b)^2}{P_a^2})} \tag{0.23}$$

5. We're given P_a^{α} and P_b^{α} in some frame. We want to determine the relative velocity of the two particles in the CM frame. The magnitude of v_{rel} is

$$|v_{rel}| = |\vec{v}_a - \vec{v}_b| \tag{0.24}$$

where $\vec{v}$ in the CM frame is $\vec{v} = \frac{\vec{p'}}{E'}$

The total four momentum is

$$P^{\alpha} = P_a^{\alpha} + P_b^{\alpha} \tag{0.25}$$

The CM energy is

$$s = (P^{\alpha})^2 = m_a^2 + m_b^2 + P_a \cdot P_b \tag{0.26}$$

Using Projectors we can define the CM four vectors

$$P_a^{\prime\alpha} = P^\alpha - \frac{(P \cdot P_a)}{s} P^\alpha \tag{0.27}$$

$$P_b^{\prime\alpha} = P^\alpha - \frac{(P \cdot P_b)}{s} P^\alpha \tag{0.28}$$

Then, we have for the CM energies

$$E_a^\prime = \frac{P_a \cdot P}{\sqrt{s}} = \frac{m_a^2 + P_a \cdot P_b}{\sqrt{s}} \tag{0.29}$$

$$E_b^\prime = \frac{P_b \cdot P}{\sqrt{s}} = \frac{m_b^2 + P_a \cdot P_b}{\sqrt{s}} \tag{0.30}$$

Finally, after a little algebra,

$$|v_{rel}| = \frac{s[(P_a \cdot P_b)^2 - m_a^2 m_b^2]^{1/2}}{(m_a^2 + P_a \cdot P_b)(m_b^2 + P_a \cdot P_b)} \tag{0.31}$$

6. The Lorentz transformation is a tensor, $\mathcal{L}^{\alpha\beta}$, which transforms some four vector p^α observed by an observer moving with four velocity u^α to a vector $p^{\prime\alpha}$ as determined by an observermoving with four-velocity $u^{\prime\alpha}$.

$$\mathcal{L}^\alpha_\beta p^\beta = p^{\prime\alpha}$$

Since $\mathcal{L}$ is a tensor it must be of the form,

$$\mathcal{L}^{\alpha\beta} = A u^\alpha u^{\prime\beta} + B u^{\prime\alpha} u^\beta + C u^\alpha u^\beta + D u^{\prime\alpha} u^{\prime\beta} + E g^{\alpha\beta}$$

(a) The first part of the problem asks us to determine the coefficients A through E of the lorentz transformation tensor. By comparing the two matrix forms of the lorentz transformation, you can set up five equations relating A, B, C, D, E, v, and γ. They are as follows:

$$\begin{aligned} A\gamma + B\gamma + C + D\gamma^2 + E &= \gamma & (0.32)\\ -A\gamma v - D\gamma^2 v &= -\gamma v & (0.33)\\ B\gamma v + D\gamma^2 v &= -\gamma v & (0.34)\\ -D\gamma^2 v^2 + E &= \gamma & (0.35)\\ E &= 1 & (0.36)\\ & & (0.37) \end{aligned}$$

This system of equations has five equations and 5 unknowns, so it is possible to solve for A, B, C, D, and E in terms of γ and v. To help simplify these expressions, the equality $\gamma^2 - v^2\gamma^2 = 1$ was used. The constants were found to be:

$$\begin{aligned} A &= \frac{2\gamma+1}{\gamma+1} \quad (0.38) \\ B = C = D &= \frac{-1}{\gamma+1} \quad (0.39) \\ E &= 1 \quad (0.40) \end{aligned}$$

(b) Next we are asked to show that $\text{Ł}^{\alpha\beta}u'_\beta = u^\alpha$

By plugging in the given Lorentz transformation matrix and four vector representation and using again that $\gamma^2 - v^2\gamma^2 = 1$, we see that:

$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2 v^2 \\ -\gamma^2 v + \gamma^2 v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = u^\alpha \quad (0.41)$$

(c) Lastly, we are asked to show that $\text{Ł}^{\alpha\beta}\text{Ł}_{\beta\gamma} = g^\alpha_\gamma$, which is the identity matrix.

$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g^\beta_\gamma \quad (0.42)$$

Chapter 2

1. Consider the case for the complex scalar field interacting with a vector field, see Eq. (2.5). Find an expression for the current, j^α, in terms of ϕ, derivatives, and eA, such that one satisfies the equation of continuity, $d_\alpha j^\alpha = 0$.

   Consider the current density

$$j^\alpha = \frac{1}{2}\big[\phi^*(id^\alpha - eA^\alpha)\phi + \phi(-id^\alpha - eA^\alpha)\phi^*\big].$$

This equation is obtained by taking the real part of the current density $\phi^* i d^\alpha \phi$ while accounting for the potentials eA^α. To ensure that the continuity equation is satisfied, we take the derivative of j^α:

$$\begin{aligned}
d_\alpha j^\alpha =& \frac{1}{2}\big[d_\alpha(\phi^* i d^\alpha \phi) - d_\alpha(\phi^* eA^\alpha\phi) - d_\alpha(\phi i d^\alpha \phi^*) - d_\alpha(\phi eA^\alpha \phi^*)\big] && (0.43)\\
=& \frac{i}{2}\big[(d_\alpha\phi^*)(d^\alpha\phi) + \phi^* d_\alpha d^\alpha \phi + \phi i e A^\alpha d_\alpha \phi^* + \phi^* i d_\alpha(eA^\alpha\phi) \\
& - (d^\alpha\phi^*)(d_\alpha\phi) - \phi d_\alpha \phi^\alpha \phi^* + \phi^* i e A^\alpha d_\alpha\phi + \phi i d_\alpha(eA^\alpha\phi^*)\big] && (0.44)\\
=& \frac{i}{2}\big[(d_\alpha\phi^*)(d^\alpha\phi) + \phi^*(d^2\phi + i d_\alpha(eA^\alpha\phi) + ieA^\alpha d_\alpha\phi) \\
& - (d_\alpha\phi^*)(d_\alpha\phi) + \phi(-d^2\phi^* + i d_\alpha(eA^\alpha\phi^*) + ieA^\alpha d_\alpha\phi^*)\big] && (0.45)\\
=& \frac{i}{2}\big[\phi^*(d^2\phi + i d_\alpha(eA^\alpha\phi) + ieA^\alpha d_\alpha\phi) \\
& + \phi(-d^2\phi^* + i d_\alpha(eA^\alpha\phi^*) + ieA^\alpha d_\alpha\phi^*)\big]. && (0.46)
\end{aligned}$$

The Klein-Gordon equation allows to to rewrite Eq. (**??**) as

$$d_\alpha j^\alpha = \frac{i}{2}\big[\phi^*(e^2A^2\phi - m^2\phi) + \phi(m^2\phi^* - e^2A^2\phi^*)\big],$$

which is simply

$$d_\alpha j^\alpha = 0. \qquad \square$$

2. Consider the case of a particle of momentum $p_x > 0$ incident on a step function potential as was done in the example above, only with the potential being the zeroth component of a vector potential,

$$eA_0 = \left\{ \begin{array}{rl} 0, & x < 0 \\ V_0, & x > 0 \end{array} \right. .$$

Let the mass, m, be constant and let $A_i(x) = 0$ everywhere.

(a) Calculate the transmission and reflection coefficients.

(b) Show that current is conserved across the boundary.

(c) What range(s) of V_0 allow for propagating solutions (not exponentially damped) for $x > 0$?

(d) Calculate and plot both the charge density j_0 and current density j_x as functions of x for one example in each separate range in (c). Be sure to plot for both positive and negative x.

(a)

We assume a solution of the form

$$\psi(x) = \left\{ \begin{array}{rl} e^{ip_x x} + Be^{-ip_x x}, & x < 0 \\ Ce^{ip'_x x}, & x > 0 \end{array} \right. .$$

The boundary conditions require the wave function and its first derivative to be continuous at $x = 0$, giving

$$1 + B = C \tag{0.47}$$

$$ip_x - ip_x B = ip'_x C, \tag{0.48}$$

which can be solved algebraically to give B and C as

$$B = \frac{p_x - p'_x}{p_x + p'_x} \tag{0.49}$$

$$C = \frac{2p_x}{p_x + p'_x}. \tag{0.50}$$

The reflection coefficient is then

$$R = |B|^2 = \left(\frac{p_x - p'_x}{p_x + p'_x} \right)^2,$$

while the transmission coefficient is given as

$$T = 1 - R = \frac{4p_x p'_x}{(p_x + p'_x)^2}.$$

(b)

Current density is given by

$$j^\alpha = \big[(d^\alpha \psi^*)\psi - \psi^*(d^\alpha \psi)\big].$$

For $\psi(x)$ as given in Eq. (**??**), we consider first the region $x < 0$. Then, we have

$$\begin{aligned}
j^\alpha_{x<0} =& \frac{i}{2}\big[(-ip_x e^{-ip_x} + iB^* p_x e^{ip_x})(e^i p_x x + Be^{-ip_x x}) \\
& - (e^{-ip_x x} + B^* e^{ip_x x})(ip_x e^{-ip_x x} - ip_x B e^{-ip_x x})\big] \qquad (0.51) \\
=& \frac{i}{2}\big[- ip_x - ip_x B e^{-2ip_x x} + iB^* p_x e^{2ip_x x} + ip_x|B|^2 \\
& - ip_x + ip_x B e^{-2ip_x x} - ip_x B^* e^{2ip_x x} + ip_x |B|^2\big] \qquad (0.52) \\
=& \frac{i}{2}\big(- 2ip_x + 2ip_x|B|^2\big) \qquad (0.53) \\
=& p_x\big(1 - |B|^2\big) \qquad (0.54) \\
=& \frac{4p_x^2 p'_x}{(p_x + p'_x)^2}. \qquad (0.55)
\end{aligned}$$

Similarly, for the region $x > 0$, we have

$$\begin{aligned}
j^\alpha_{x>0} =& \frac{i}{2}\big(- ip'_x|C|^2 - ip'_x|C|^2\big) \qquad (0.56) \\
=& p'_x|C|^2 \qquad (0.57) \\
=& \frac{4p_x^2 p'_x}{(p_x + p'_x)^2}. \qquad (0.58)
\end{aligned}$$

Thus, current is conserved across the boundary at $x = 0$ since

$$j^\alpha_{x<0} = j^\alpha_{x>0}. \qquad \square$$

(c)

We know from energy conservation that

$$E = \sqrt{p_x^2 + m^2} = \sqrt{p_x'^2 + m^2} + V_0.$$

To avoid exponentially damped solutions when $x > 0$, we choose p_x' to be real so that the exponent in $\psi(x > 0) = e^{ip_x' x}$ is complex. The limiting values of V_0 that correspond to p_x' being real are found by setting p_x' equal to zero and solving for V_0, that is,

$$\begin{aligned} p_x' =& \sqrt{(E - V_0)^2 - m^2} = 0 && (0.59) \\ E - V_0 =& \pm m && (0.60) \\ V_0 =& E \pm m. && (0.61) \end{aligned}$$

Thus, the ranges of V_0 for which solutions propagate are $V_0 \geq E + m$ and $V_0 \leq E - m$.

(d)

Charge density j_0 is given by

$$j_0 = \begin{cases} \frac{i}{2}[\psi^* d_t\psi - \psi d_t\psi^*], & x < 0 \\ \frac{1}{2}[\psi^*(id_t - V_0)\psi + \psi(-id_t - V_0)\psi^*], & x > 0 \end{cases}$$

while current density j_x is represented as

$$j_x = \frac{i}{2}\big[(d_x\psi^*)\psi - \psi^*(d_x\psi)\big]$$

for

$$\psi(x,t) = \begin{cases} e^{i(Et - p_x x)} + Be^{-i(Et - p_x x)}, & x < 0 \\ Ce^{i(Et - p_x' x)}, & x > 0 \end{cases}.$$

Note that for this solution p_x and p_x' are the three-dimensional momenta for the two regions $x < 0$ and $x > 0$, respectively. Solving for j_0 and j_t for both regions yields

$$\begin{aligned} j_0(x<0) =& -E\big(1 + |B|^2\big) = -2\sqrt{p_x^2 + m^2}\frac{(p_x^2 + p_x'^2)}{(p_x + p_x')^2} && (0.62) \\ j_0(x>0) =& (E - V_0)|C|^2 = 4p_x^2\frac{\sqrt{p_x^2 + m^2} - V_0}{(p_x + p_x')^2} && (0.63) \\ j_t(x<0) =& p_x(|B|^2 - 1) = -\frac{4p_x^2 p_x'}{(p_x + p_x')^2)} && (0.64) \\ j_t(x>0) =& -p_x'|C|^2 = -\frac{4p_x^2 p_x'}{(p_x + p_x')^2)}. && (0.65) \end{aligned}$$

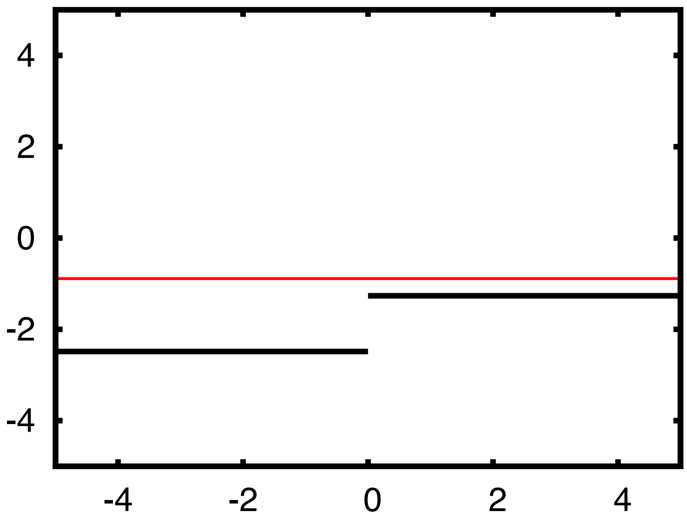

Figure 0.1: Plot showing j_0 and j_x for $V_0 > E + m$.

We see that j_x is conserved across the boundary $x = 0$, whereas j_0 is not conserved. Plots were made of the charge and current densities. For all plots, we let $p_x = 1$, $p'_x = 2$, and $m = 2$, while we set $V_0 = 1.2(E+m)$ in Fig. (**??**) and $V_0 = 1.2(E-m)$ in Fig. (**??**). While not shown explicitly in the plots, note that when $V_0 = E + m$ that $j_0 = j_x$ and when $V_0 = E - m$ that $j_0 = -j_x$.

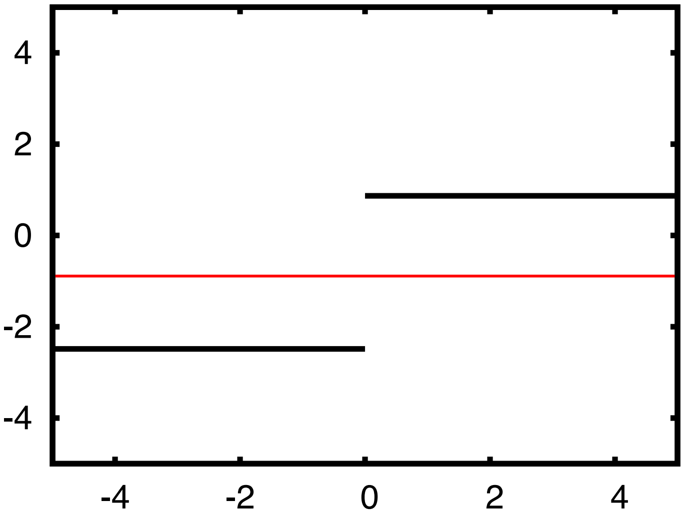

Figure 0.2: Plot showing j_0 and j_x for $V_0 < E + m$

3. Consider the Langrangian density for the electromagnetic field, $A^\mu(x)$, coupled to an external current $j(x)$:

$$\text{Ł} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e j \cdot A, \qquad F_{\mu\nu} \equiv d_\mu A_\nu - d_\nu A_\mu .$$

(a) Solve for the equations of motion for A^μ and show that this is equivalent to four of Maxwell's equations. Derive and express the four equations in terms of $\vec{E}$ and $\vec{B}$.

(b) The other four equations come from the identity $d_\alpha \tilde{F}^{\alpha\beta} = 0$, where $\tilde{F}^{\alpha\beta} \equiv \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$. Since $\epsilon^{\alpha\beta\gamma\delta}$ is anti-symmetric in the indices, and since d_μ appears twice in $d_\alpha \tilde{F}^{\alpha\beta}$,

$$d_\alpha \tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} d_\alpha (d_\gamma A_\delta - d_\delta A_\gamma) = 0.$$

Express these equations in terms of $\vec{E}$ and $\vec{B}$.

Chapter 3

1. (a) First, let's calculate β^2, $\{\alpha^i, \beta\}$ and $\{\alpha^i, \alpha^j\}$

$$\beta^2 = \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

$$\{\alpha^i, \beta\} = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = 0$$

$$\{\alpha^i, \alpha^j\} = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} + \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} = 2\delta^{ij}$$

Now, we expand write the square as two different terms.

$$(p^i\alpha^i + m\beta)^2 = (p^i\alpha^i + m\beta)(p^j\alpha^j + m\beta)$$

Then by multiplying the terms out we get:

$$= p^i\alpha^i p^j\alpha^j + p^i\alpha^j m\beta + m\beta p^j\alpha^j + m^2\beta^2$$

Remembering that $\beta^2 = 1$ and that $\{\beta, \alpha^i\} = 0$ we obtain:

$$= p^i p^j \alpha^i \alpha^j + p^i m\alpha^i\beta - p^j m\alpha^j\beta + m^2$$

We can rewrite the first term as:

$$p^i p^j \alpha^i \alpha^j = 1/2 p^i p^j (\alpha^i\alpha^j + \alpha^j\alpha^i) = 1/2 p^i p^j \{\alpha^i, \alpha^j\} = p^i p^j \delta^{ij}$$

We can rewrite it in the above form since we are summing over i and j. We now obtain the desired result:

$$(p^i\alpha^i + m\beta)^2 = |\vec{p}|^2 + m^2$$

(b) $u(\mathbf{p})$ and $v(\mathbf{p})$ are solutions to the free Dirac equation. Guess solution of the form:

$$u(\mathbf{p}) = \begin{pmatrix} \phi(\mathbf{p}) \\ \chi(\mathbf{p}) \end{pmatrix} e^{-ip\cdot x}$$

where ϕ and χ are two component spinors. Since $\mathbf{p}$ is only along the positive z-axis, $p \cdot x = pz$. Plugging into the Dirac Equation in the chiral representation, we get:

$$Eu(\mathbf{p}) = (\mathbf{p} \cdot \boldsymbol{\alpha})u(\mathbf{p}) + m\beta u(\mathbf{p})$$

Plugging in the α and β matrices in the chiral representation, and remembering only α^z contributes, we get:

$$E \begin{pmatrix} \phi(\mathbf{p}) \\ \chi(\mathbf{p}) \end{pmatrix} = p^z \begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{pmatrix} + m \begin{pmatrix} 0 & -\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

Multiplying the matrices out, we obtain:

$$E\phi = p\sigma^z\phi - m\chi \tag{0.66}$$

and

$$E\chi = p\sigma^z\chi - m\phi$$

We want $\phi(\mathbf{p})$ to be an eigenfunction of $\mathbf{p}\cdot\boldsymbol{\sigma}$ so:

$$p\sigma^z\phi_\pm = \pm p\phi_\pm \Rightarrow \phi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \phi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Plugging ϕ into (**??**), we obtain an equation for $u(\mathbf{p})$. These solutions are:

$$\chi_\pm = \frac{1}{m}(E \mp p)\phi_\pm$$

$$\Rightarrow u_+(\mathbf{p}) = \begin{pmatrix}1\\0\\\frac{E-p}{m}\\0\end{pmatrix} \qquad u_-(\mathbf{p}) = \begin{pmatrix}0\\1\\0\\\frac{E+p}{m}\end{pmatrix}$$

The same process can be used to find the equation $v_\pm(\mathbf{p})$ with $E \to -E$, this results in:

$$v_+(\mathbf{p}) = \begin{pmatrix}1\\0\\-\frac{E+p}{m}\\0\end{pmatrix} \qquad v_-(\mathbf{p}) = \begin{pmatrix}0\\1\\0\\\frac{-E+p}{m}\end{pmatrix}$$

Normalizing the solutions, such that $\bar{u}u = 2E$, one obtains the solutions:

$$u_+\mathbf{p} = \frac{\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u_-(\mathbf{p}) = \frac{\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

And doing the same for $\bar{v}v$:

$$v_+(\mathbf{p}) = \frac{-\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \qquad v_-(\mathbf{p}) = \frac{-\not{p} + m}{\sqrt{2(E+m)}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

(c) First, let us look at the limit $p \to 0$. This implies that $E \to m$, giving the solutions for u:

$$u_+(\mathbf{p}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u_-(\mathbf{p}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

And the solutions for v:

$$v_+(\mathbf{p}) = \sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \qquad v_-(\mathbf{p}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

In the limit that $p \to \infty$, we obtain the solutions for u:

$$u_+(\mathbf{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad u_-(\mathbf{p}) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

And the solutions for v:

$$v_+(\mathbf{p}) = \sqrt{2E}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad v_-(\mathbf{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

(d) In the massless limit, the Dirac Equation simplifies to:

$$Eu(\mathbf{p}) = (\boldsymbol{\alpha}\cdot\mathbf{p})u(\mathbf{p})$$

Also, in the massless limit, $E = |\mathbf{p}|$

$$\Rightarrow \pm|\mathbf{p}|\phi_\pm = (\mathbf{p}\cdot\boldsymbol{\sigma})\phi_\pm \quad , \quad \pm|\mathbf{p}|\chi_\pm = -(\mathbf{p}\cdot\boldsymbol{\sigma})\chi_\pm$$

Also, we want $(\mathbf{p}\cdot\boldsymbol{\Sigma})u = +|\mathbf{p}|u$, applying the two conditions leads to the forms for the solutions being:

$$u(\mathbf{p})_\pm = \begin{pmatrix}\phi_\pm\\0\end{pmatrix} \quad , \quad v(\mathbf{p})_\pm \begin{pmatrix}0\\\phi_\pm\end{pmatrix}$$

Solving for $\phi_\pm$, one obtains:

$$\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}\phi_\pm = \pm|\mathbf{p}|\phi_\pm$$

This leads to the solutions. The solution for positive helicity and positive $|\mathbf{p}|$ is given by:

$$u(\mathbf{p}) = \begin{pmatrix}p_z + |\mathbf{p}|\\ p_x + ip_y\\ 0\\ 0\end{pmatrix}$$

The solution for negative helicity and positive $|\mathbf{p}|$ is given by:

$$u(\mathbf{p}) = \begin{pmatrix}p_z - |\mathbf{p}|\\ p_x + ip_y\\ 0\\ 0\end{pmatrix}$$

The solution for positive helicity and negative $|\mathbf{p}|$ is given by:

$$v(\mathbf{p}) = \begin{pmatrix}0\\ 0\\ p_z + |\mathbf{p}|\\ p_x + ip_y\end{pmatrix}$$

The solution for positive helicity and negative $|\mathbf{p}|$ is given by:

$$v(\mathbf{p}) = \begin{pmatrix} 0 \\ 0 \\ p_z - |\mathbf{p}| \\ p_x + ip_y \end{pmatrix}$$

(e) If one only used two-by-two matrices, then the matrices in the Dirac Equation would be the Pauli Matrices. In this situation, the solutions would be spinors instead of bi-spinors. Also, this would result in having the positive energy solutions always having positive helicity, and negative energy solutions always having negative helicity.

2. Show that $\not{p}^2 = p^2$.

Note that $\not{p}$ is a 4×4 matrix and so the p^2 on the right hand side is multiplied by an implied 4×4 identity.

$$\begin{aligned} \not{p}^2 &= (\gamma_\mu p^\mu)^2 && (0.67) \\ &= \gamma_\mu p^\mu \gamma_\nu p^\nu && (0.68) \\ &= \frac{1}{2}\{\gamma_\mu, \gamma^\nu\} p^\mu p_\nu && (0.69) \end{aligned}$$

Now, using $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}\hat{I}$.

$$\begin{aligned} \not{p}^2 &= \frac{1}{2} 2 g_\mu{}^\nu \hat{I} p^\mu p_\nu && (0.70) \\ &= p^\mu p_\mu && (0.71) \\ &= p^2 && (0.72) \end{aligned}$$

3. Show that $v(-\vec{p}) = (\vec{\alpha} \cdot \hat{p}) u(\vec{p})$. I.e., show that if u is a positive energy solution, $(\vec{\alpha} \cdot \vec{\hat{p}}) u(\vec{p})$ will be a negative energy solution.

u and v are the positive and negative plane wave solutions to the Dirac equation. Let's choose the solutions to be oriented along the z-axis in the chiral representation as we found in problem 1.

$$u(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{E-p}{m} \\ 0 \end{pmatrix}, \quad v(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{-E-p}{m} \\ 0 \end{pmatrix} \qquad (0.73)$$

Then

$$
\begin{aligned}
(\vec{\alpha}\cdot\hat{p})u(\vec{p}) &= \begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}\cdot\hat{p}_z\begin{pmatrix}1\\0\\\frac{E-p}{m}\\0\end{pmatrix} && (0.74)\\
&= \begin{pmatrix}\sigma_z & 0\\ 0 & -\sigma_z\end{pmatrix}\begin{pmatrix}1\\0\\\frac{E-p}{m}\\0\end{pmatrix} && (0.75)\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\\frac{E-p}{m}\\0\end{pmatrix} && (0.76)\\
&= \begin{pmatrix}1\\0\\\frac{-E+p}{m}\\0\end{pmatrix} && (0.77)\\
&= v(-\vec{p}) && (0.78)
\end{aligned}
$$

4. Show that $(\not{p}+m)/2m$ is a projection operator that projects out either the positive or negative energy solutions for momentum $\mathbf{p}$ depending on whether p_0 is $+E_p$ or $-E_p$.

   A projection operator acting on itself, should yield itself.Using our result from Problem 2, $\not{p}^2 = p^2 = m^2$ get see that

$$\left(\frac{\not{p}+m}{2m}\right)^2 = \frac{(\not{p}^2 + 2m\not{p} + m^2)}{4m^2} = \frac{2m^2 + 2m\not{p}}{4m^2} = \frac{\not{p}+m}{2m} \tag{0.79}$$

   So $(\not{p}+m)/2m$ is a projection operator.

   The most general solution to the Dirac equation is a linear combination of the positive energy solution and the negative energy solution $\psi = Au(p) + Bv(p)$. The projection operator should project ψ onto $u(p)$ if $p_0 = +E_p$ and ψ onto $v(p)$ if $p_0 = -E_p$ where explicitly

$$(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma} - m)u(p_0,\vec{p}) = 0 \tag{0.80a}$$
$$(-p_0\gamma_0 + \vec{p}\cdot\vec{\gamma} - m)v(p_0,\vec{p}) = 0 \tag{0.80b}$$

   or

$$(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma})u(p_0,\vec{p}) = mu(p_0,\vec{p}) \tag{0.81a}$$
$$(-p_0\gamma_0 + \vec{p}\cdot\vec{\gamma})v(p_0,\vec{p}) = mv(p_0,\vec{p}) \tag{0.81b}$$

If $p_0 = E_p$ then,

$$\left(\frac{\not{p}+m}{2m}\right)\psi = \frac{A}{2m}\left(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma}+m\right)u + \frac{B}{2m}\left(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma}+m\right)v \tag{0.82}$$

$$= \frac{A}{2m}(m+m)u + \frac{B}{2m}(-m+m)v = Au \tag{0.83}$$

Conversely if $p_0 = -E_p$ then,

$$\left(\frac{\not{p}+m}{2m}\right)\psi = \frac{A}{2m}\left(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma}+m\right)u + \frac{B}{2m}\left(p_0\gamma_0 - \vec{p}\cdot\vec{\gamma}+m\right)v \tag{0.84}$$

$$= \frac{A}{2m}(-m+m)u + \frac{B}{2m}(m+m)v = Bv \tag{0.85}$$

5. Beginning with calculating $u = Su_0$:

$$S = e^{\gamma^0\gamma^3\frac{\eta}{2}}$$

$$S = \cosh(\frac{\eta}{2}) - \sinh(\frac{\eta}{2})\gamma^0\gamma^3$$

$$S = \begin{pmatrix} \cosh(\frac{\eta}{2}) & 0 & -\sinh(\frac{\eta}{2}) & 0 \\ 0 & \cosh(\frac{\eta}{2}) & 0 & \sinh(\frac{\eta}{2}) \\ -\sinh(\frac{\eta}{2}) & 0 & \cosh(\frac{\eta}{2}) & 0 \\ 0 & \sinh(\frac{\eta}{2}) & 0 & \cosh(\frac{\eta}{2}) \end{pmatrix}$$

Acting S on u_0, where $u_0 = (1,0,0,0)$ gives:

$$Su_0 = \begin{pmatrix} \cosh(\frac{\eta}{2}) \\ 0 \\ -\sinh(\frac{\eta}{2}) \\ 0 \end{pmatrix} \tag{0.86}$$

Now calculating $u(\mathbf{p}) = \frac{1}{Z(\mathbf{p})}(\not{p}+m)u_0$:

$$Z(\mathbf{p})u(\mathbf{p}) = (\not{p}+m)u_0$$

$$\Rightarrow u(\mathbf{p}) = \frac{1}{Z(\mathbf{p})}\begin{pmatrix} p_0+m & 0 & -|\mathbf{p}| & 0 \\ 0 & p_0+m & 0 & |\mathbf{p}| \\ |\mathbf{p}| & 0 & -p_0+m & 0 \\ 0 & -|\mathbf{p}| & 0 & -p_0+m \end{pmatrix}$$

$$\Rightarrow u(\mathbf{p}) = \frac{1}{Z(\mathbf{p})}\begin{pmatrix} p_0+m \\ 0 \\ |\mathbf{p}| \\ 0 \end{pmatrix} \tag{0.87}$$

Setting each term in (**??**) and (**??**) equal to each other, one obtains:

$$Z(\mathbf{p})\cosh(\frac{\eta}{2}) = p_0 + m \quad , \quad -Z(\mathbf{p})\sinh(\frac{\eta}{2}) = |\mathbf{p}|$$

Squaring both equations and subtracting the second one from the first, one obtains:

$$(Z(\mathbf{p}))^2(\cosh^2(\frac{\eta}{2}) - \sinh^2(\frac{\eta}{2})) = (p_0 + m)^2 - |\mathbf{p}|^2 = p_0^2 + 2p_0 m + m^2 - |\mathbf{p}|^2 = 2m(p_0 + m)$$

Solving for $Z(\mathbf{p})$ one obtains:

$$Z(\mathbf{p}) = \sqrt{2m(p_0 + m)}$$

Chapter 4

1. The Dirac Equation in free space is

$$\hat{H} = \mathbf{p} \cdot \boldsymbol{\alpha} + \hat{\beta} m$$

In the presence of a magnetic field the momentum becomes

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A} \text{ where } \mathbf{p} = \sum p_i \hat{e}_i \text{ and } p_i = -i\delta_i$$

This implies that the hamiltonian takes the form

$$\hat{H} = (\mathbf{p} - e\mathbf{A}) \cdot \boldsymbol{\alpha}$$

The conditions that $A_x = A_z = A_0 = 0$, $A_y = Bx$, and $\boldsymbol{\alpha} = \vec{\sigma}$ make the equation

$$\hat{H} = p_x \sigma_x + p_z \sigma_z + (p_y - eBx)\,\sigma_y$$

$$= -i\delta_x \sigma_x - i\delta_z \sigma_z + (-i\delta_y - eBx)\,\sigma_y$$

(B) Now we'll try a solution of the form

$$\psi = e^{-iEt + ik_y y + ik_z z} e^{-\frac{(x-x_0)^2}{2R^2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To do this, apply the hamiltonian to ψ and equate it with $E\psi$. First, the hamiltonian will be written in a more convenient form.

$$\hat{H} = p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (p_y - eBx) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} p_z & p_x - i\,(p_y - eBx) \\ p_x + i\,(p_y - eBx) & -p_z \end{pmatrix}$$

$$\hat{H}\psi = \begin{pmatrix} p_z & p_x - i\,(p_y - eBx) \\ p_x + i\,(p_y - eBx) & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iEt + ik_y y + ik_z z} e^{-\frac{(x-x_0)^2}{2R^2}}$$

$$\hat{H}\psi = \begin{pmatrix} p_z \\ p_x + i\,(p_y - eBx) \end{pmatrix} e^{-iEt + ik_y y + ik_z z} e^{-\frac{(x-x_0)^2}{2R^2}}$$

$$\hat{H}\psi = \begin{pmatrix} -i\frac{\delta}{\delta z} \\ -i\frac{\delta}{\delta x} + i\left(-i\frac{\delta}{\delta y} - eBx\right) \end{pmatrix} e^{-iEt + ik_y y + ik_z z} e^{-\frac{(x-x_0)^2}{2R^2}}$$

$$\Rightarrow \hat{H}\psi = \begin{pmatrix} k_z \\ \frac{i(x-x_0)}{R^2} + i\,(k_y - eBx) \end{pmatrix} e^{-iEt + ik_y y + ik_z z} e^{-\frac{(x-x_0)^2}{2R^2}} = E\psi$$

From the form of ψ, we know that:

$$k_z = E$$

$$0 = \frac{x - x_0}{R^2} + k_y - eBx$$

$$\Rightarrow k_y = eBx - \frac{x - x_0}{R^2}$$

However, k_y should not depend on x so we should choose R and x_0 such that any x dependence disappears.

$$k_y = \left(eB - \frac{1}{R^2} \right) x + \frac{x_0}{R^2}$$

$$\Rightarrow R^2 = \frac{1}{eB} \text{ and } x_0 = \frac{k_y}{eB}$$

So the eigen-energy is zero for a particle with no motion in the z-direction and the case of motion in the z-direction, the energy is the momentum in the z-direction.

Chapter 5

5.1

Consider the non-relativistic case presented in Sec. 5.1. Consider a state at $t=0$ whose state is a wave packet of the form

$$|\boldsymbol{k}_0,\kappa\rangle = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{(2\pi\kappa^2)^{3/4}} d^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/4\kappa^2} a^\dagger(\boldsymbol{k})|0\rangle.$$

(a) Calculate $\langle \boldsymbol{k}_1,\kappa\rangle \boldsymbol{k}_2,\kappa$.

(b) Calculate $\langle \boldsymbol{k}_0,\kappa|H\rangle \boldsymbol{k},\kappa$, where H is given in Eq. (5.12).

Starting from the given expression for $|\boldsymbol{k}_0,\kappa\rangle$, we have

$$\begin{aligned}
\langle \boldsymbol{k}_1,\kappa\rangle \boldsymbol{k}_2,\kappa =& \frac{1}{(2\pi)^3}\int \frac{d^3k}{(2\pi\kappa^2)^{3/4}}\int \frac{d^3k'}{(2\pi\kappa^2)^{3/4}} e^{-(\boldsymbol{k}-\boldsymbol{k}_1)^2/4\kappa^2} \\
&\times e^{-(\boldsymbol{k}'-\boldsymbol{k}_2)^2/4\kappa^2}\langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{k}')\rangle 0 && (0.88)\\
=& \frac{1}{(2\pi)^3}\int \frac{d^3k}{(2\pi\kappa^2)^{3/4}}\int \frac{d^3k'}{(2\pi\kappa^2)^{3/4}} e^{-(\boldsymbol{k}-\boldsymbol{k}_1)^2/4\kappa^2} \\
&\times e^{-(\boldsymbol{k}'-\boldsymbol{k}_2)^2/4\kappa^2}\delta^3(\boldsymbol{k}-\boldsymbol{k}') && (0.89)\\
=& \frac{1}{(2\pi)^3(2\pi\kappa)^{3/2}}\int d^3k\; e^{-(\boldsymbol{k}-\boldsymbol{k}_1)^2/4\kappa^2-(\boldsymbol{k}-\boldsymbol{k}_2)/4\kappa^2} && (0.90)\\
=& \frac{e^{-[\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]/4\kappa^2}}{(2\pi)^3(2\pi\kappa)^{3/2}}\int d^3k\; e^{-[\boldsymbol{k}-(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]/2\kappa^2} && (0.91)\\
=& \frac{e^{-[\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]/4\kappa^2}}{(2\pi)^3\cancel{(2\pi\kappa)^{3/2}}}\cancel{(2\pi\kappa)^{3/2}}, && (0.92)
\end{aligned}$$

resulting in

$$\langle \boldsymbol{k}_1,\kappa\rangle \boldsymbol{k}_2,\kappa = \frac{e^{-[\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+(\boldsymbol{k}_1+\boldsymbol{k}_2)^2]/4\kappa^2}}{(2\pi)^3}.$$

(b)

The Hamiltonian given in Eq. 5.12 is

$$H = \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{2m}a^\dagger(\boldsymbol{k})a(\boldsymbol{k}).$$

From this definition, we have

$$
\begin{aligned}
\langle \boldsymbol{k}_0, \kappa | H \rangle \boldsymbol{k}_0, \kappa =& \frac{1}{2m(2\pi)^6(2\pi\kappa^2)^{3/2}} \int d^3k \int d^3k' \int d^3k'' \boldsymbol{k}''^2 e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/4\kappa^2} \\
& \times e^{-(\boldsymbol{k}'-\boldsymbol{k}_0)^2/4\kappa^2} \langle 0 | a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}'') a(\boldsymbol{k}'') a^\dagger(\boldsymbol{k}') \rangle 0 \tag{0.93} \\
=& \frac{1}{2m(2\pi)^6(2\pi\kappa^2)^{3/2}} \int d^3k \int d^3k' \int d^3k'' \boldsymbol{k}''^2 e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/4\kappa^2} \\
& \times e^{-(\boldsymbol{k}'-\boldsymbol{k}_0)^2/4\kappa^2} (2\pi)^3 \delta^3(\boldsymbol{k}'-\boldsymbol{k}'') \delta^3(\boldsymbol{k}-\boldsymbol{k}'') \tag{0.94} \\
=& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \int d^3k \ \boldsymbol{k}^2 e^{-(\boldsymbol{k}-\boldsymbol{k}_0)^2/2\kappa^2} \tag{0.95} \\
=& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \left[\int dk \ k^2 e^{-(k-k_0)^2/2\kappa^2} \right]^3 . \tag{0.96} \\
& \tag{0.97}
\end{aligned}
$$

If we let $\chi = k - k_0$, we can rewrite the integral as

$$
\begin{aligned}
\langle \boldsymbol{k}_0, \kappa | H \rangle \boldsymbol{k}_0, \kappa =& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \left[\int d\chi \ (\chi + k_0)^2 e^{-\chi^2/2\kappa^2} \right]^3 \tag{0.98} \\
=& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \left[\int d\chi \ (\chi^2 + \cancelto{\text{odd}}{2\chi k_0} + k_0^2) e^{-\chi^2/2\kappa^2} \right]^3 \tag{0.99} \\
=& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \left[\int d\chi \ \chi^2 e^{-\chi^2/2\kappa^2} + k_0^2 \int d\chi \ e^{-\chi^2/2\kappa^2} \right]^3 \tag{0.100} \\
=& \frac{1}{2m(2\pi)^3(2\pi\kappa^2)^{3/2}} \left[\frac{1}{2} \left[\pi (2\kappa^2)^3 \right]^{1/2} + k_0^2 (2\pi\kappa^2)^{1/2} \right]^3 \tag{0.101} \\
=& \frac{1}{2m(2\pi)^3\cancel{(2\pi\kappa^2)^{3/2}}} \cancel{(2\pi\kappa^2)^{3/2}} (k_0^2 + \kappa^2)^3 , \tag{0.102}
\end{aligned}
$$

resulting in

$$
\langle \boldsymbol{k}_0, \kappa | H \rangle \boldsymbol{k}_0, \kappa = \frac{1}{2m} \left(\frac{k_0^2 + \kappa^2}{2\pi} \right)^3 .
$$

Consider a transformation to a frame moving with rapidity η along the x-axis. For such a boost the Lorentz transformation matrix is

$$L^alpha_\beta = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the Jacobian for the Lorentz transform from d^4x to d^4x'.

The Jacobian is simply the determinant of the L^alpha_β, giving us

$$\begin{aligned} \mathcal{J} &= \det L && (0.103) \\ &= \begin{vmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} && (0.104) \\ &= \cosh^2\eta - \sinh^2\eta, && (0.105) \end{aligned}$$

which is simply

$$\mathcal{J} = 1.$$

Express an eigenstate of the number operator, $|m\rangle$, in terms of coherent states, i.e., find $g_m(\eta)$ in the expression below:

$$|m\rangle = \int d\eta_r d\eta_i g_m(\eta)|\eta\rangle.$$

By the completeness relation, we can write

$$\begin{aligned}
|m\rangle &= \int \frac{d\eta_r d\eta_i}{\pi}|\eta\rangle\langle\eta\rangle m \qquad (0.106)\\
&= \int \frac{d\eta_r d\eta_i}{\pi}|\eta\rangle e^{-|\eta|^2/2}\langle 0|e^{\eta^* a}\rangle m \qquad (0.107)\\
&= \int \frac{d\eta_r d\eta_i}{\pi}|\eta\rangle e^{-|\eta|^2/2}\langle 0|\sum_{n=0}^{\infty}\frac{(\eta^* a)^n}{n!}\rangle m \qquad (0.108)\\
&= \int \frac{d\eta_r d\eta_i}{\pi}|\eta\rangle e^{-|\eta|^2/2}\langle m|\sqrt{m!}\frac{(\eta^*)^m}{m!}\rangle m \qquad (0.109)\\
&= \int d\eta_r d\eta_i \frac{(\eta^*)^m}{\pi\sqrt{m!}} e^{-|\eta|^2/2}|\eta\rangle, \qquad (0.110)
\end{aligned}$$

giving

$$g_m(\eta) = \frac{(\eta^*)^m}{\pi\sqrt{m!}}e^{-|\eta|^2/2}.$$

Consider the Hamiltonian derived at the end of 5.6

$$\begin{aligned} H &= \int d^3r \big[\pi(\boldsymbol{r})\dot{\phi}(\boldsymbol{r}) - \text{Ł}\big] \qquad (0.111) \\ &= \int \frac{d^3r}{2}\big[d_t\phi(\boldsymbol{r},t)\phi(\boldsymbol{r},t) + \nabla\phi(\boldsymbol{r})\cdot\phi(\boldsymbol{r}) + m^2\phi(\boldsymbol{r})^2\big]. \end{aligned}$$

Using the expressions for the field operators in terms of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$, derive an expression for H as an integral over momentum with the creation and destruction operators in the integrand.

Let us first write ϕ, $\nabla\phi$, and π in terms of a and $a^\dagger$. We have

$$\phi = \int \widetilde{dp}\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) + e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big] \qquad (0.112)$$

$$\nabla\phi = -\,i\int \widetilde{dp}\;\boldsymbol{p}\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) - e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big] \qquad (0.113)$$

$$\pi = -\,i\int \widetilde{dp}\;E_{\boldsymbol{p}}\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) - e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big], \qquad (0.114)$$

where we define the shorthand

$$\widetilde{dp} = \frac{d^3p}{2E_{\boldsymbol{p}}(2\pi^3)}.$$

The Hamiltonian can then be expressed as

$$H = \frac{1}{2}\int d^3r \int \widetilde{dp} \int \widetilde{dk}$$

$$\times\Bigg\{E_{\boldsymbol{p}}E_{\boldsymbol{k}}i^2\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) - e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big]\Big[e^{i\boldsymbol{k}\cdot\boldsymbol{r}}a(\boldsymbol{k}) - e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{k})\Big]$$

$$+ (-i)^2(\boldsymbol{p}\cdot\boldsymbol{k})\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) - e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big]\Big[e^{i\boldsymbol{k}\cdot\boldsymbol{r}}a(\boldsymbol{k}) -$$

$$+ m^2\Big[e^{i\boldsymbol{p}\cdot\boldsymbol{r}}a(\boldsymbol{p}) + e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}a^\dagger(\boldsymbol{p})\Big]\Big[e^{i\boldsymbol{k}\cdot\boldsymbol{r}}a(\boldsymbol{k}) + e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}a^\dagger$$

$$(0.115)$$

$$
\begin{aligned}
H =& \frac{1}{2}\int d^3r \int \widetilde{dp} \int \widetilde{dk} \\
& \times \Bigg\{ \Big(- E_{\boldsymbol{p}}E_{\boldsymbol{k}} - \boldsymbol{p}\cdot\boldsymbol{k} \Big) \Big[e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a(\boldsymbol{k}) - e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a^\dagger(\boldsymbol{k}) \\
& - e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}} a^\dagger(\boldsymbol{p})a(\boldsymbol{k}) + e^{-i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{k}) \Big] \\
& + m^2 \Big[e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a(\boldsymbol{k}) + e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a^\dagger(\boldsymbol{k}) \\
& + e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a(\boldsymbol{k}) + e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a(\boldsymbol{k}) \Big] \Bigg\} \qquad (0.116) \\
=& \frac{1}{2}\int d^3r \int \widetilde{dp} \int \widetilde{dk} \\
& \times \Bigg\{ \Big(- E_{\boldsymbol{p}}E_{\boldsymbol{k}} - \boldsymbol{p}\cdot\boldsymbol{k} + m^2 \Big) \Big[e^{i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a(\boldsymbol{k}) + e^{-i(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{r}} a^\dagger(\boldsymbol{p})a^\dagger(\boldsymbol{k}) \Big] \\
& + \Big(E_{\boldsymbol{p}}E_{\boldsymbol{k}} + \boldsymbol{p}\cdot\boldsymbol{k} + m^2 \Big) \Big[e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{r}} a(\boldsymbol{p})a^\dagger(\boldsymbol{k}) + e^{i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{r}} a^\dagger(\boldsymbol{p})a(\boldsymbol{k}) \Big] \Bigg\}. \qquad (0.117)
\end{aligned}
$$

Since we know the identity

$$
\int d^3r \; e^{\pm i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = (2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}'),
$$

we can simply assume the Hamiltonian to be

$$
\begin{aligned}
H =& \frac{1}{4E_{\boldsymbol{p}}} \int \widetilde{dp} \Bigg\{ \Big(\cancelto{0}{- E_{\boldsymbol{p}}^2 + |\boldsymbol{p}|^2 + m^2} \Big) \Big[a(\boldsymbol{p})a(\text{-}\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a^\dagger(\text{-}\boldsymbol{p}) \Big] \\
& + \Big(E_{\boldsymbol{p}}^2 + \cancelto{E_{\boldsymbol{p}}^2}{|\boldsymbol{p}|^2 + m^2} \Big) \Big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) \Big] \Bigg\} \qquad (0.118) \\
=& \frac{1}{2} \int \widetilde{dp} \; E_{\boldsymbol{p}} \Big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + 2E_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}) \Big] \qquad (0.119) \\
=& \int \widetilde{dp} \; E_{\boldsymbol{p}} \Big[a(\boldsymbol{p})a^\dagger(\boldsymbol{p}) + E_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}) \Big]. \qquad (0.120)
\end{aligned}
$$

The last term in the expression for the Hamiltonian can be dropped due to consideration of the vacuum energy, leaving us with a final expression for the Hamiltonian, given as

$$
H = \int \widetilde{dp} \; E_{\boldsymbol{p}} a(\boldsymbol{p}) a^\dagger(\boldsymbol{p}).
$$

Chapter 6

6.1

We are given

$$u_s\left(\mathbf{p}\right) = \big(\not{p} + m\big)\, u_s\left(\mathbf{p} = \mathbf{0}\right)/\sqrt{2m(E_p+m)} \tag{0.121}$$

and

$$v_s\left(\mathbf{p}\right) = (-\not{p} + m)v_s(\mathbf{p} = \mathbf{0})/\sqrt{2m(E_p+m)} \tag{0.122}$$

We will use these to show the following four identities.
It will be useful to define the adjoints of the above expressions for use later.

$$u_s^\dagger(\mathbf{p}) = u_s^\dagger(\mathbf{p} = \mathbf{0})(\not{p}^\dagger + m))/\sqrt{2m(E_p+m)} \tag{0.123}$$

$$v_s^\dagger(\mathbf{p}) = v_s^\dagger(\mathbf{p} = \mathbf{0})(-\not{p}^\dagger + m)/\sqrt{2m(E_p+m)} \tag{0.124}$$

Also it will be useful to recall the expressions for $u_s^+(0)$, $u_s^-(0)$, and $v_s^+(0)$, $v_s^-(0)$

1a: Show $\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = \delta_{ss'}$

$$\begin{aligned}
\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) &= u_s^\dagger(\mathbf{p})\gamma^0 u_{s'}(\mathbf{p}) \\
&= \frac{1}{2m(E_p+m)}\big(u_s^\dagger(\mathbf{0})(\not{p}^\dagger + m)\gamma^0(\not{p}+m)u_{s'}(\mathbf{0})\big) \\
&= \frac{1}{2m(E_p+m)}\big(u_s^\dagger(\mathbf{0})(\not{p}^\dagger\gamma^0 + m\gamma^0)(\not{p}+m)u_{s'}(\mathbf{0})\big) \\
&= \frac{1}{2m(E_p+m)}\big(u_s^\dagger(\mathbf{0})(\gamma^0\not{p} + \gamma^0 m)(\not{p}+m)u_{s'}(\mathbf{0})\big) \\
&= \frac{1}{2m(E_p+m)}\big(\bar{u}_s(\mathbf{0})(\not{p}+m)(\not{p}+m)u_{s'}(\mathbf{0})\big) \\
&= \frac{1}{2m(E_p+m)}\big(\bar{u}_s(\mathbf{0})(2m\not{p}+2m^2)u_{s'}(\mathbf{0})\big) \\
&= \frac{1}{2m(E_p+m)}\big(\bar{u}_s(\mathbf{0})(2mE_p u_{s'}(\mathbf{0}) + 2m^2 u_{s'}(\mathbf{0}))\big) \\
&= \frac{1}{2m(E_p+m)}(2m(E_p+m))(\bar{u}_s(\mathbf{0})u_{s'}(\mathbf{0})) \\
&= \delta_{ss'}
\end{aligned}$$

1b: Show $\bar{u}_s(\mathbf{p})v_{s'}(\mathbf{p}) = 0$

$$\begin{aligned}
\bar{u}_s(\mathbf{p})v_{s'}(\mathbf{p}) &= u_s^\dagger(\mathbf{p})\gamma^0 v_{s'}(\mathbf{p}) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger + m)\gamma^0(-\not{p}+m)v_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger\gamma^0 + m\gamma^0)(-\not{p}+m)v_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\gamma^0\not{p} + \gamma^0 m)(-\not{p}+m)v_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(\bar{u}_s(\mathbf{0})(-\not{p}^2 + m^2)v_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(\bar{u}_s(\mathbf{0})(-m^2 + m^2)v_{s'}(\mathbf{0})\right) \\
&= 0
\end{aligned}$$

1c: Show $u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = \frac{E_p}{m}\delta_{ss'}$

$$\begin{aligned}
u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) &= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger + m)(\not{p}+m)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger + m)(\not{p}+m)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger\not{p} + m\not{p}^\dagger + m\not{p} + m^2)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger\not{p} + 2m\gamma_0 E_p + m^2)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})\left(E_p^2 + 2(\gamma\cdot\mathbf{p}) + \mathbf{p}^2 + 2m\gamma_0 E_p + m^2\right)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})\left(2E_p^2 + 2(\gamma\cdot\mathbf{p}) + 2m\gamma_0 E_p\right)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}(2E_p^2 + 2mE_p)(u_s^\dagger(\mathbf{0})u_{s'}(\mathbf{0})) \\
&= \left(\frac{E_p}{m}\right)\delta_{ss'}
\end{aligned}$$

$$\begin{aligned}
\not{p}^\dagger\not{p} &= \left(\gamma_0 E_p + \gamma\cdot\mathbf{p}\right)\left(\gamma_0 E_p - \gamma\cdot\mathbf{p}\right) \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + (\gamma\cdot\mathbf{p})^2 \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + (\gamma_i p^i)^2 \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + \gamma_i\gamma_j p^i p^j \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + \frac{1}{2}\left(\gamma_i\gamma_j + \gamma_j\gamma_i\right) p^i p^j \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + \frac{1}{2}\{\gamma_i, \gamma_j\} p^i p^j \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + \delta_{ij} p^i p^j \\
&= E_p^2 + 2(\gamma\cdot\mathbf{p}) + \mathbf{p}^2
\end{aligned}$$

1d: Show $u_s^\dagger(\mathbf{p})v_{s'}(-\mathbf{p}) = 0$

$$\begin{aligned}
u_s^\dagger(\mathbf{p})v_{s'}(-\mathbf{p}) &= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(\not{p}^\dagger + m)(-\not{p}^\dagger + m)v_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(-\not{p}^\dagger\not{p}^\dagger + \not{p}^\dagger m - m\not{p}^\dagger + m^2)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(-(\not{p}\not{p})^\dagger + m^2)u_{s'}(\mathbf{0})\right) \\
&= \frac{1}{2m(E_p+m)}\left(u_s^\dagger(\mathbf{0})(-m^2 + m^2)u_{s'}(\mathbf{0})\right) \\
&= 0
\end{aligned}$$

6.2

Recall a couple of things

$$\psi^\dagger(\alpha\cdot\mathbf{p} + \gamma_0 m)\psi = \psi^\dagger i\partial_0\psi \tag{0.125}$$

$$\mathcal{L} = \bar{\psi}i\not{\partial}\psi - m\bar{\psi}\psi \tag{0.126}$$

$$\psi = \sum_s \int \frac{md^3p}{(2\pi)^3 E_p}(b_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}) \tag{0.127}$$

$$\psi^\dagger = \sum_s \int \frac{md^3p}{(2\pi)^3 E_p}(u_s^\dagger(\mathbf{p})b_s^\dagger(\mathbf{p})e^{ip\cdot x} + v_s^\dagger(\mathbf{p})d_s(\mathbf{p})e^{-ip\cdot x}) \tag{0.128}$$

$$
\begin{aligned}
T_{00} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\psi)}(\partial_0\psi) - g_{00}\mathcal{L} \\
&= (\bar{\psi}i\gamma_0)(\partial_0\psi) - g_{00}\mathcal{L} \\
&= (\bar{\psi}i\gamma_0\partial^0\psi) - (\bar{\psi}i\gamma_0\partial^0\psi - \bar{\psi}i\gamma_i\partial^i\psi - m\bar{\psi}\psi) \\
&= (\bar{\psi}i\gamma\cdot\nabla\psi) + m\bar{\psi}\psi \\
&= (\bar{\psi}(i\gamma\cdot\nabla + m)\psi \\
&= \psi^\dagger\gamma_0(i\gamma\cdot\nabla + m)\psi \\
&= \psi^\dagger(\alpha\cdot\mathbf{p} + \gamma_0 m)\psi \\
&= \psi^\dagger i\partial_0\psi \\
&= \psi^\dagger i\partial_0\left(\sum_s\int\frac{md^3p}{(2\pi)^3E_p}(b_s(\mathbf{p})e^{-ip\cdot x}u_s(\mathbf{p}) + d_s^\dagger(\mathbf{p})e^{ip\cdot x}v_s(\mathbf{p}))\right) \\
&= \psi^\dagger\left(\sum_s\int\frac{md^3p}{(2\pi)^3}(b_s(\mathbf{p})e^{-ip\cdot x}u_s(\mathbf{p}) - d_s^\dagger(\mathbf{p})e^{ip\cdot x}v_s(\mathbf{p}))\right) \\
&= \sum_{ss'}\int\int\frac{m^2d^3pd^3p'}{(2\pi)^6E_{p'}}(u_{s'}^\dagger(\mathbf{p}')b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})u_s(\mathbf{p})e^{i(\mathbf{p}'-\mathbf{p})\cdot x} - v_{s'}^\dagger(\mathbf{p}')d_s(\mathbf{p})d_{s'}^\dagger(\mathbf{p}')v_s(\mathbf{p})e^{i(\mathbf{p}-\mathbf{p}')\cdot x} + cts)
\end{aligned}
$$

$$
\begin{aligned}
H &= \int d^3r T_{00} \\
&= \int d^3r\sum_{ss'}\int\int\frac{m^2d^3pd^3p'}{(2\pi)^6E_{p'}}(u_{s'}^\dagger(\mathbf{p}')b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})u_s(\mathbf{p})e^{i(\mathbf{p}'-\mathbf{p})\cdot x} - v_{s'}^\dagger(\mathbf{p}')d_s(\mathbf{p})d_{s'}^\dagger(\mathbf{p}')v_s(\mathbf{p})e^{i(\mathbf{p}-\mathbf{p}')\cdot x} + cts) \\
&= \sum_{ss'}\int\int\frac{m^2d^3pd^3p'}{(2\pi)^6E_{p'}}(u_{s'}^\dagger(\mathbf{p}')b_{s'}^\dagger(\mathbf{p}')b_s(\mathbf{p})u_s(\mathbf{p})(2\pi)^3\delta(\mathbf{p}'-\mathbf{p}) - v_{s'}^\dagger(\mathbf{p}')d_s(\mathbf{p})d_{s'}^\dagger(\mathbf{p}')v_s(\mathbf{p})(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')) \\
&= \sum_{ss'}\int\frac{m^2d^3p}{(2\pi)^3E_p}(b_{s'}^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s(\mathbf{p})d_{s'}^\dagger(\mathbf{p})) \\
&= \sum_{s}\int\frac{m^2d^3p}{(2\pi)^3E_p}(b_{s}^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s(\mathbf{p})d_{s}^\dagger(\mathbf{p}))
\end{aligned}
$$

6.3

Consider the quantity $\bar{u}_s(\mathbf{p})\,v_{s'}(-\mathbf{p})$. Terms like this arise in the case where a scalar field ϕ might couple to particle-antiparticle production. Assuming $\mathbf{p}$ lies along the z axis, calculate the overlap. Express your answer in terms of E_p, m and p_z.

Using the identities given in problem 1 we can write

$$\bar{u}_s\left(\mathbf{p}\right)v_{s'}\left(-\mathbf{p}\right) = \frac{u_s^\dagger\left(0\right)\left(\not{p}^\dagger + m\right)\gamma_0\left(-\not{p}^\dagger + m\right)v_{s'}\left(0\right)}{2m\left(E_p + m\right)} \tag{0.129}$$

Then, realizing that $\not{p}^\dagger\gamma_0 = \gamma_0\not{p}$,

$$\begin{aligned}\bar{u}_s\left(\mathbf{p}\right)v_{s'}\left(-\mathbf{p}\right) &= \frac{u_s^\dagger\left(0\right)\gamma_0\left(-\not{p}\not{p}^\dagger - m\not{p}^\dagger + m\not{p} + m^2\right)v_{s'}\left(0\right)}{2m\left(E_p + m\right)} &(0.130)\\ &= \frac{u_s^\dagger\left(0\right)\gamma_0\left(\left(-E_p^2 - p_z^2 - 2p_zE_p\gamma_0\gamma_3\right) + m^2\right)v_{s'}\left(0\right)}{2m\left(E_p + m\right)} &(0.131)\\ &= \frac{u_s^\dagger\left(0\right)\left(-2\gamma_0p_z^2 - 2p_zE_p\gamma_3\right)v_{s'}\left(0\right)}{2m\left(E_p + m\right)} &(0.132)\end{aligned}$$

To get to **??** we've used that $\mathbf{p} = p_z\hat{\mathbf{z}}$, then that $E_p^2 = p_z^2 + m^2$ to get to **??**. You can show that $\gamma_0v_{s'}\left(0\right) = -v_{s'}\left(0\right)$, and since $u_{s'}\left(0\right)$ and $v_{s'}\left(0\right)$ are orthogonal, and we can further simplify the expression.

$$\bar{u}_s\left(\mathbf{p}\right)v_{s'}\left(-\mathbf{p}\right) = \frac{u_s^\dagger\left(0\right)\left(-p_zE_p\gamma_3\right)v_{s'}\left(0\right)}{m\left(E_p + m\right)} \tag{0.133}$$

Finally we can write out the matrices in the Dirac representation to show that

$$\bar{u}_s\left(\mathbf{p}\right)v_{s'}\left(-\mathbf{p}\right) = \frac{-p_zE_p}{m\left(E_p + m\right)}\begin{pmatrix}(1,0) & (0,1) & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ (1,0)\\ (0,1)\end{pmatrix} \tag{0.134}$$

where the parenthetical values are determined by whether spin is up or down. Therefore

$$\bar{u}_s\left(\mathbf{p}\right)v_{s'}\left(-\mathbf{p}\right) = \mp\frac{p_zE_p}{m\left(E_p + m\right)} \tag{0.135}$$

For both spins up or down respectively, or 0 if the spins are different.

6.4

Consider the quantity $\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right)$. Terms like this arise in the case where a vector field A^μ might couple to particle-antiparticle production. Assuming $\mathbf{p}$ lies along the z axis, calculate the overlap for all four values of μ, and for all combinations of s and s', where the spin labels refer to various eigenstates of Σ_z. Scott if you read this, I'll buy you a drink next time you come to happy hour -Matt. Express your answer in terms of E_p, m and p_z.

If $\gamma^\mu = \gamma^0$, then using the identity proved in problem 1 part d.

$$\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right) = u_s^\dagger(\mathbf{p})\gamma^0\gamma^0v_{s'}(\mathbf{p}) = 0 \tag{0.136}$$

If $\gamma^\mu = \gamma_i$, then

$$\begin{aligned}
\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right) &= \frac{u_s^\dagger(0)(\not{p}^\dagger + m)\gamma^0\gamma^i(-\not{p}^\dagger + m)v_{s'}(0)}{2m(E_p + m)} && (0.137)\\
&= \frac{\bar{u}_s(0)(\not{p}\not{p}^\dagger - m\not{p} + m\not{p}^\dagger + m^2)\gamma^i v_{s'}}{2m(E_p + m)} && (0.138)\\
&= \frac{\bar{u}_s(0)(E_p^2 + 2\gamma^0\gamma^3 P_z E_p + p_z^2 + 2m\gamma^0 E_p + m^2)\gamma^i v_{s'}(0)}{2m(E_p + m)} && (0.139)
\end{aligned}$$

Using $\gamma^0 = (\gamma^0)^\dagger \Rightarrow u_s^\dagger(0)\gamma^0 = \left(\gamma^0 u_s(0)\right)^\dagger = u_s^\dagger(0)$,

$$\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right) = \frac{u^\dagger(0)(2E_p^2\gamma^i + 2E_p p_z\gamma^3\gamma^i + 2mE_p\gamma^i)v_{s'}(0)}{2m(E_p + m)} \tag{0.140}$$

Note that any $\gamma^i\gamma^j$ term cannot contribute since the product will be a block diagonal matrix which won't allow the lower components in the $v(0)$ to dot into the upper components of $u(0)$. So

$$\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right) = \frac{2E_p(E_p + m)}{2m(E_p + m)} u_s^\dagger(0)\gamma^i v_{s'}(0) \tag{0.141}$$

If you're hardcore you can choose your favorite representation and calculate $u_s^\dagger(0)\gamma^i v_{s'}(0)$ by hand for each gamma and spin combination, or you can run it through the computer like we did.

$$\bar{u}_s\left(\mathbf{p}\right)\gamma^\mu v_{s'}\left(-\mathbf{p}\right) = \frac{E_p}{m} A_{mn} \tag{0.142}$$

and

$$A_{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \tag{0.143}$$

where $m = \mu$ of the γ^μ and n runs over u_+v_+, u_+v_- ,u_-v_+, u_-v_- respectively.

Chapter 7

7.1

$$G(x) = -iT < 0|\Phi(x)\Phi^\dagger(0)|0 >,$$

$$\Phi(x) = \int \frac{d^3k}{2E(2\pi)^3}[a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x}], [a(k), a^\dagger(q)] = 2E_k(2\pi)^3\delta^{(3)}(k-q)$$

$$G(x) = -i\int \frac{d^3k}{2E_k(2\pi)^3}\frac{d^3q}{2E_q(2\pi)^3}T < 0|[a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x}][a^\dagger(q) + b(q)]|0 >$$

$$= -i\int \frac{d^3k}{2E_k(2\pi)^3}\frac{d^3q}{2E_q(2\pi)^3}T < 0|a(k)a^\dagger(q)e^{-ik\cdot x}+b^\dagger(k)b(q)e^{ik\cdot x}+b(k)a^\dagger(q)e^{ik\cdot x}+a(k)b(q)e^{-ik\cdot x}|0 >$$

$$= -i\int \frac{d^3k}{2E_k(2\pi)^3}\frac{d^3q}{2E_q(2\pi)^3}T < 0|a(k)a^\dagger(q)e^{-ik\cdot x}|0 >$$

$$= -i\int \frac{d^3k}{2E_k(2\pi)^3}T < 0|a(k)a^\dagger(k)e^{-ik\cdot x}|0 >$$

$$= -i\int \frac{d^3k}{2E_k(2\pi)^3}[e^{-ik\cdot x}\Theta(x_0)\Big|_{k0=E_k} + e^{ik\cdot x}\Theta(-x_0)\Big|_{k0=E_k}]$$

Preforming contour integration in the typically manor, one obtains:

$$G(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{1}{k^2-m^2+i\epsilon}, \epsilon \to 0$$

Taking the Fourier Transform of this one obtains:

$$G(k) = \frac{1}{k^2-m^2+i\epsilon}$$

7.2

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 W_\mu)} = -\frac{1}{4}\epsilon^{\alpha\nu 0\mu}F_{\alpha\nu}$$

$$= -\frac{1}{4}\epsilon^{\alpha\nu 0\mu}\epsilon_{\alpha\nu\sigma\rho}\partial^\sigma W^\rho$$

$$= F^{0\mu}$$

Now calculate the commutation relation, $[\Pi^\mu(y), W^\mu(x)]$, Notice that since $F^{00} = 0$ only the spatial components will contribute to the commutation relation:

$$[\Pi^i(y), W^j(x)] = [\dot{W}^i(y), W^j(x)] + [\partial^i W^0(y), W^j(x)]$$

$$[\dot{W}^j(y), W^i(x)] = -i\sum_{s,r}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3 2E_q}\{-\epsilon_s^j(\mathbf{k})\epsilon_r^{i*}(\mathbf{q})e^{-ik\cdot y+iq\cdot x}[a_s(\mathbf{k}), a_r^\dagger(\mathbf{q})]$$
$$+\epsilon_s^{j*}(\mathbf{k})\epsilon_r^i(\mathbf{q})e^{ik\cdot y-iq\cdot x}[a_s^\dagger(\mathbf{k}), a_r(\mathbf{q})]\}$$

$$[\dot{W}^j(y), W^i(x)] = -i \sum_s \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \{\epsilon_s^i(\mathbf{k})\epsilon_s^{j*}(\mathbf{k}) e^{ik\cdot(x-y)} + \epsilon_s^i(\mathbf{k})\epsilon_s^{j*}(\mathbf{k}) e^{-ik\cdot(x-y)}\}$$

$$\sum \epsilon_s^i(\mathbf{k})\epsilon_s^{j*}(\mathbf{k}) = \delta^{ij} + \frac{k^i k^j}{m^2}$$

Combining these results and preforming a ton of algebra, one obtains:

$$\begin{aligned}[\Pi^i(y), W^j(x)] &= -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} (\delta^{ij}) \{e^{ik\cdot(x-y)} + e^{-ik\cdot(x-y)})\} \\ &= -i\delta^{ij}\delta^{(3)}(x-y)\end{aligned}$$