

Physics 853 Quiz #3 - Monday, Oct 5, 2011

1. Consider the matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\alpha_x\alpha_y\alpha_z.$$

- (a) Find the eigenvalues of γ_5 .
- (b) Show that $(1 + \gamma_5)/2$ is a projection operator.
- (c) Write down the matrix components of γ_5 in the chiral representation, i.e. where

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}.$$

FYI: $\epsilon_{ijk}\sigma_i\sigma_j = 2i\sigma_k$

- (d) Consider a positive-energy plane-wave solution with positive/negative helicity,

$$\begin{aligned} (\mathbf{p} \cdot \boldsymbol{\alpha})u_{\pm}(\mathbf{p}) &= E_p u_{\pm}(\mathbf{p}), \\ (\boldsymbol{\Sigma} \cdot \mathbf{p})u_{\pm}(\mathbf{p}) &= \pm E_p u_{\pm}(\mathbf{p}), \end{aligned}$$

where the spin operator $\Sigma_k = -i\epsilon_{ijk}\alpha_i\alpha_j/2$. Independent of the representation, show that for such solutions,

$$\gamma_5 u_{\pm}(\mathbf{p}) = \pm u_{\pm}(\mathbf{p}).$$

HINT: Consider $\{\mathbf{p} \cdot \boldsymbol{\alpha}, \mathbf{p} \cdot \boldsymbol{\Sigma}\}u_{\pm}(\mathbf{p})$, and you may appreciate the identity $\{\alpha^m, \epsilon_{ijk}\alpha^j\alpha^k\} = 2\delta_{im}\epsilon_{ijk}\alpha^i\alpha^j\alpha^k$ (note: don't sum over i).

2. Consider massless particles in a magnetic field, $\vec{A} = xB\hat{y}$. Within the chiral representation,

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix},$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

assume the solution has the form

$$\psi = e^{-iEt+ik_z z} \begin{pmatrix} \phi(x) \\ \chi(x) \\ 0 \\ 0 \end{pmatrix}.$$

(a) Through substitution, show that the equation for ϕ is

$$(E^2 - k_z^2)\phi(x) = (-\partial_x^2 + e^2 B^2 x^2 - eB)\phi(x).$$

(b) Solve for the eigenstate and eigen-energies assuming that ϕ has the form

$$\phi(x) = e^{-x^2/2R^2},$$

i.e., find R and E in terms of eB and k_z .

(c) By making an analogy with harmonic oscillators, find the full spectrum of energies. In other words, beginning with the non-relativistic Hamiltonian,

$$H\psi = -\frac{\partial_x^2}{2m}\psi + \frac{1}{2}m\omega_0^2 x^2\psi,$$

one knows that the eigen-energies are $\epsilon_n = (n + 1/2)\omega_0$.