Physics 853 Quiz #2 - Monday, Sep. 26, 2012

1. Consider a particle living in a one-dimensional world and interacting with the zero$^{\text{th}}$ component of a vector potential and obeying the Klein-Gordon equation,

$$(i\partial - eA)^2\phi = m^2\phi,$$

and the vector potential is $eA = (V_0, 0, 0, 0)$, with

$$V_0 = \begin{cases} 0, & x < 0 \\ 3m, & x > 0. \end{cases}$$

Let the particle incoming from the left have momentum $p_x \to 0$.

(a) Solve for solutions, i.e., find p', B and C for solutions of the form

$$\phi(x) = \begin{cases} e^{ipx} + Be^{-ipx}, & x < 0 \\ Ce^{ip' \cdot x}, & x > 0 \end{cases} .$$

(b) Which quantities are continuous across the $x = 0$ border? (circle continuous quantities)

$\phi(x)$ $\quad$ $j_0(x)$ $\quad$ $j_x(x)$

(c) What are the charge densities and current densities on each side? For the charge densities, find the average over a long distance (neglect oscillating pieces).

2. Given the Lagrangian equations of motion,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial \phi}\mathcal{L},$$

show that the stress-energy tensor,

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}\partial^\beta \phi - g^{\alpha\beta}\mathcal{L},$$

represents four conserved currents (energy current and momentum current), i.e. show that:

$$\partial_\alpha T^{\alpha\beta} = 0.$$

HINT: Remember that $\mathcal{L}$ is a function of ϕ and $\partial_\mu \phi$.

3. Consider the Klein-Gordon Lagrangian density for a complex field interacting with an external electromagnetic field

$$\mathcal{L} = \left[-i\partial^\alpha - eA^\alpha(x)\right]\phi^*(x)\left[i\partial_\alpha - eA_\alpha(x)\right]\phi(x) - m^2\phi^*(x)\phi(x).$$

Given that the stress-energy tensor is given by

$$T_{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi_k}\partial_\beta \phi_k - g_{\alpha\beta}\mathcal{L},$$

with the sum over k refers to a sum over ϕ and ϕ^*,

(a) Derive an expression for the energy density operator, T_{00}.

(b) What is the current density for a state, $\phi(x) = e^{-ip\cdot x}/\sqrt{\Omega}$, and a constant (independent of x) vector potential A_α? Also, express the eigen-energy p_0 in terms of $|\vec{p}|$ and A_α. (Ω is a volume, and "eigen-energy" refers to being a solution of the K.G. equation)

(c) What is the energy density for the same state?