Worksheet #3 – PHY102 (Spring 2011)

Matrices and linear algebra

Last week we did vector operations with *lists*. This week we introduce you to *matrices*: their representation using lists, and some of the matrix operations that can be done in Mathematica.

(This worksheet assumes that you already have some familiarity with with what a matrix is, and what it means to multiply a matrix by a vector. If this is all new to you, ask the instructor for a brief introduction!)

Let us first see how to represent matrices in Mathematica as a *list*. Type "m={{a,b}, {c,d}}". Now type "MatrixForm[m]". Note that all of the Mathematica operations must be applied to the *list* form of the matrix (not the matrix form). You can see that you get the matrix m with its elements a, b, c, d in the usual form. You can think of this matrix as consisting of two row vectors (a,b) and (c,d). Type "m[[1]]" and check you get the first row vector (a,b). Now type "m[[1,1]]"; this will give you the 1st element of the first vector, namely a (notice that "m[[1]][[1]]" also does the same thing). Likewise, to access the element d, type "m[[2,2]]". To access the element c, type "m[[2,1]]". Note that the element i, j in m[[i,j]] corresponds directly to the subscripts in the mathematical matrix form $m_{i,j}$—the first index is the row and the second index is the column. As you have done with vectors, you can perform algebraic operations on matrices. You can multiply a matrix with a vector. To see this, type "r={x,y}". In order to take a dot product of the matrix m with this vector r, Type "m.r" (or Dot[m,r]). Now type "Dimensions[m]". The output (2,2) verifies that the matrix m is a 2×2 matrix.

At times, you need to get the *transpose* of a matrix, which is obtained by exchanging its off-diagonal elements (in this case the elements c and d). Type "t=Transpose[m]". Now type "MatrixForm[t]". You see that the matrix has diagonal elements the same but the elements c and d got interchanged with respect to the original matrix m. In general, that transpose is defined mathematically by $\widetilde{M}_{i,j} = M_{j,i}$.

Often we require the *determinant* of a matrix, which is a scalar quantity constructed from the elements. Type “Det[m]” which will give you the *determinant* of the matrix m. Now type “Det[t]” and verify that the *determinant* is the same for the transposed matrix. A *diagonal matrix* has all off-diagonal elements set to zero. Type “DiagonalMatrix[{e,f}]”. Now type “MatrixForm[DiagonalMatrix[{e,f}]]”.
The “Inverse” of a matrix is the matrix which, when multiplied by the original matrix, produces a unit diagonal matrix (unit matrix, as it is often called). Type “mi=Inverse[m]”. Now take the product “h=m.mi” and check that h is indeed a unit matrix (you may have to perform “Simplify” on h). In general any $n \times n$ matrix has n eigenvalues and n eigenvectors. Type “Eigenvalues[m]” followed by “Eigenvectors[m]” to see what they are for the matrix m.

Assignment 3

Problem 1. A 2×2 matrix A is constructed from the following rows: (5,3) and (2,1).
(i) Write it in matrix form.
(ii) Find its determinant.
(iii) Find its transpose.
(iv) Find its inverse A^{-1}.
(v) Check $A{\cdot}A^{-1}$ is a unit matrix.
(vi) Check $A^{-1}{\cdot}A$ is also a unit matrix.
(vii) Find its eigenvalues and eigenvectors.
(viii) Check that $A \cdot$ eigenvector $=$ eigenvalue times eigenvector for each of the eigenvectors.

Problem 2.
(i) Find the solution to the following set of equations:

$$2x - y + 2z = 2$$

$$-x + 5y + z = 1$$

$$2x + y + 6z = 1$$

by writing them as a matrix equation $\mathbf{Av} = \mathbf{b}$, whose solution is $\mathbf{v} = \mathbf{A}^{-1}\mathbf{b}$. (This is not the most straightforward way to solve these equations using Mathematica—see below—but it is the method you would have learned when you first learned matrices.)

Check that you have the solution by evaluating $A \cdot v - b$.

Also solve these equations using Mathematica's `Solve` command. (To do that, define the solution vector in terms of its components, e.g., `u = {ux, uy, uz}`, and note that the equation to be solved must be typed with a double equal sign, e.g. `A.u - b == 0`.

Also solve these equations using Mathematica's `LinearSolve` command.

(ii) Try to find the solution to the following set of equations:

$$1x + 2y - z = -0.4$$

$$1.3x - 3.2y + 1.3z = 1$$

$$-2.5x - 5y + 2.5z = 1$$

What does Mathematica tell you? What does this mean? From a mathematical point of view, explain why this set of equations does not have a unique solution. Write your answers in a text cell.

Also solve these equations using the `Solve` command.

Also solve these equations using the `LinearSolve` command, and note that it is misleading, since it gives only one out of the infinite number of possible solutions!