Worksheet #7 – PHY102 (Spring 2011)

Collisions

In this worksheet, we will return to solving equations and solving differential equations.

Often there are multiple ways to accomplish something in Mathematica. Usually one way is easier than another but less elegant. Why might you want to use the elegant method rather than the "easy" one? Because it can often save trouble later on in your Mathematica session. Here is an example. Let's say you want to know the distance a mass of 50 kg falls in 30 s, after falling out of an airplane. Obviously you want to use $y = v_0 t + \frac{1}{2}at^2$, where $v_0 = 0$, $t = 30\,\mathrm{s}$ and $a = -g = -9.81\,\mathrm{m/s^2}$. The simplest way is to type directly into Mathematica:

```
y = -9.81*30^2/2
```

A more elegant route is to type:

```
v0 = 0;
a = -9.81;
t = 30;
y = v0*t + a*t^2/2
```

Or a more space-saving way would be:

```
{v0,a,t}={0,-9.81,30};
y=v0*t + a*t^2/2
```

A problem with these approaches may arise later, because you have permanently defined the variables `v0`, `a` and `t` to these values, so wherever they appear later in your Mathematica notebook, those numerical values will be substituted—possibly leading to undesired results. It is to clean up messes like this that we often use

```
Remove["Global`*"]
```

Furthermore, whenever you are exploring a physics problem, you will often find yourself wanting to see what happens if you change the initial assumptions, such as the values of `v0`, `a`, and `t` in this example.

A more elegant solution to this problem is to define the equation in algebraic form, and then obtain the particular solution using substitutions. One way to do this is to write

```
y = v0*t + a*t^2/2
sol1 = y /. {v0 → 0, a → -9.81, t → 30}
```

(To make the → symbol, just type "-" followed by ">" and Mathematica will automatically produce the desired arrow.)

Or similarly, you can write

```
y[t_] := v0*t + a*t^2/2
sol1 = y[t] /. {v0 → 0, a → -9.81, t → 30}
```

In this method, you could also replace the second line by

```
sol1 = y[30] /. {v0 → 0, a → -9.81}
```

These solutions are elegant because the quantity of interest is defined as a function, so we can operate on it (for example, find its derivative etc.). We found the particular solution we were looking for (i.e., got the same result as we did when we used the "easy" methods shown above) but didn't permanently reset the values of any internal variables in Mathematica. Try it. Enter the different commands above. Then check what Mathematica thinks the variables (e.g., a and v0) are after each case. Use a `Remove["Global`*"]` in between each test. You may have already been using this "substitution" technique, if you have been using the **DSolve** example given out with worksheet 4. When you use **DSolve** or **Solve**, for example, the solutions to the equation are returned as a list of substitutions. You can see this by entering the following code:

```
(* This solves two simultaneous linear equations *)
    f1[x_,y_]:= a*x + b*y + c
    f2[x_,y_]:= d*x + e*y + f
    sol = Solve[{f1[x,y]==0, f2[x,y]==0}, {x,y}]
    {x,y}={x,y}/. sol[[1]]

(* This checks to see that the solutions are correct *)
    Simplify[f1[x,y]]
    Simplify[f2[x,y]]
```

The definition `f1[x_,y_]:=`... defines a function whose two arguments can later be called anything you like. It is often simpler to specify a function in the following way, which is equivalent to the above:

```
(* Alternative solution for the two simultaneous equations *)
    Remove["Global‘*"]
    f1 = a*x + b*y - c ;
    f2 = c*x + d*y - e ;
    sol = Solve[f1 == 0, f2 == 0, x, y]
    x, y = x, y /. sol[[1]]

(*This checks to see that the solutions are correct*)
    Simplify[f1]
    Simplify[f2]
```

Problem 1

Use Mathematica to solve the following problem (using the examples above to see how to solve the equations). A ball of mass **m** moving horizontally with a velocity **ui** undergoes a head-on *elastic* collision with another ball of mass **M** traveling at velocity **Ui**. Apply conservation of momentum and energy to find expressions for the final velocities **uf** and **Uf** of these two particles as a function of **m**, **M**, **ui** and **Ui**. Verify your solutions by confirming that they preserve energy and momentum conservation.

Now find the final velocities for each of the following special cases:

(i) $\mathbf{m} = \mathbf{M}$
(ii) $\mathbf{m} = \mathbf{2M}$
(iii) $\mathbf{m} = \mathbf{0}$

Problem 2

A particle of mass m traveling with speed v in the horizontal direction strikes a pendulum, which consists of a thin uniform rod of length A and mass M which is initially hanging vertically at rest. The particle hits the very bottom of the pendulum and sticks to it there. As a result, the center of mass of the pendulum+particle system rises to a maximum vertical distance H above its original value. After that, if falls back and continues to oscillate forever since we ignore friction.

Note that Energy is not conserved during the collision: the collision is inelastic or the projectile wouldn't stick. Also Momentum is not conserved, because the pivot point of the pendulum supplies a force.

The quantity that is conserved during the collision is the angular momentum, so that is the conservation law you need to find the initial conditions at t = 0, where the angle of the pendulum is 0. After you use angular momentum conservation to find the initial angular velocity of the rod+projectile system, you can find the kinetic energy of that system. Then use energy conservation to compute the subsequent motion. Use the angle of the pendulum with respect to the downward direction as the coordinate.

For anyone rusty on their mechanics knowledge—or who has not yet encountered this material in other physics classes, the moment of inertia of the uniform rod pivoted about its end is $M\,A^2/3$. (You might enjoy using Mathematica to derive that result.) The moment of inertia contributed by the original projectile is $m\,A^2$ since all of that mass is located at radius A. The angular momentum of the mass+rod system is given by $L = I\,\omega$. The kinetic energy of the mass+rod system is given by $KE = (1/2)\,I\,\omega^2$. Here, $\omega = d\theta/dt$ and I is the total moment of inertia. **Note, you must choose some other name for the moment of inertia in your Mathematic code, because Mathematica insists on using I to denote the imaginary number $\sqrt{-1}$.**

(i) Find the maximum angle by which the pendulum swings (as a function of H).

(ii) Find the initial speed of the particle (as a function of H).

(iii) Now suppose $m = 1\,\text{kg}$, $M = 9\,\text{kg}$, $g = 10.0\text{m/s}^2$, $H = 4\,\text{m}$, $A = 10\,\text{m}$. Find and plot the pendulum angle as a function of time.

There are two ways to find equations of motion for this problem: you can write a differential equation for the motion using the torque due to gravity on the system; or you can use energy conservation. The energy conservation method is better, because it leads to a first-order differential equation instead of second order; and because it lets you build in the boundary condition corresponding to the initial condition of known total energy directly.

Note: when your instructor solved this problem, Mathematica was not able to solve the equation when the initial condition theta[0]==0 was included in DSolve. But without that condition, it did give a solution, which contained an unknown integration constant. You can set that constant to zero by `/. C[1] -> 0`.

(iv) Now solve the linear pendulum problem using the same parameters. (The "linear" approximation for the pendulum corresponds to approximating $\sin\theta$ by θ in the torque equation; or equivalently approximating $\cos\theta$ by $1 - \theta^2/2$ in the potential energy.)

Plot the time dependent oscillations of the full solution and the linear approximation on the same graph. Can you explain the qualitative difference between the two on physical grounds?