Worksheet #8 – PHY102 (Spring 2011)

Motion in a potential

Although you first learn about Newton's second law $\vec{F} = m\vec{a}$ and the dynamics that results from it, much of the discussion in the more advanced physics texts is in terms of "potentials" $V(\vec{r})$. A particle undergoes motion "in a potential". Note that V is a *scalar*, while $\vec{F}$ is a vector. This simplicity often makes it easier to work with the potential—unless you are dealing with problems such as those involving friction, where a potential does not exist. Visualizing the potential can also be very helpful in developing physical insight into the trajectories of the motion. The potential is also useful in understanding thermodynamic processes, which are statistical in nature.

For our purposes, we just need to know how to relate the force to the potential, and that is via the equation:

$$\vec{F}(x, y, z) \;=\; -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \,. \tag{1}$$

Often it is easier to work in polar coordinates (r, θ, ϕ)—especially when we are dealing with *central potentials*, where $V(r)$ does not depend on the angles θ and ϕ).

$$\vec{F}(r) \;=\; -\frac{\partial V}{\partial r}\,\hat{r} \qquad \text{(for central potential).} \tag{2}$$

Many of the problems you will study in undergraduate physics (and even in graduate-level physics courses) involve central potentials.

This week we study the motions in two different central potentials: the gravitational potential generated by a fixed mass M:

$$V_G(r) \;=\; -\frac{GM}{r} \,, \tag{3}$$

and the "Lennard-Jones" potential, which is an approximation to the interaction potential between two atoms of an inert gas:

$$V_{LJ}(r) \;=\; \frac{A}{r^{12}} - \frac{B}{r^{6}} \,. \tag{4}$$

(The constants A and B depend on the inert gas, *e.g.*, they are different for Helium than for Xenon.)

Problem 1.

(i) Make a plot of the Lennard-Jones potential as a function of r. For the time being, use units where $A = B = 1$.

(ii) Find the value, r_0, at which the Lennard-Jones Potential is a minimum. Find the value of the potential at its minumum: $V_{LJ}(r_0)$.

(iii) By expanding around the minimum of the Lennard-Jones potential (use the **Series** function and the **Normal** function), show that the LJ potential can be approximated near its minimum by a harmonic oscillator potential.

(iv) Make a plot of the approximate potential used in part (iii) on top of a copy of your plot of the full L-J potential from part (i).

(v) Optional challenge: find the frequency of oscillations in that approximate potential, in units where $A = B = M = 1$, with M the mass of one of the atoms. The evaluate that frequency in standard (SI) units for a pair of Argon atoms, given that the LJ potential can be written as

$$V_{LJ}(r) = 4\,\epsilon\left((\sigma/r)^{12} - (\sigma/r)^6\right) \tag{5}$$

where $\sigma = 0.36\,\mathrm{nm}$ and $\epsilon = 1.6 \times 10^{-21}\,\mathrm{J}$.

Problem 2.

(i) Make a plot of the gravitational potential energy as a function of r.

(ii) Write a piece of Mathematica code to study the motion of a comet as it approaches the sun (ignoring the effects of all of the planets!). Sun mass= $1.991 \times 10^{30}\,\mathrm{kg}$, Sun radius $= 6.96 \times 10^8\,\mathrm{m}$. Assume that the ratio (mass of comet/mass of sun) $\to$ zero, so you can take the sun to be at rest.

For a few initial conditions, plot out the trajectory of the comet as it passes by the sun.

Find some initial conditions that lead to the comet hitting the sun.

Find some initial conditions that make the comet's orbit a circle.

In case you have not yet studied mechanics at the level needed for this problem—or have forgotten it—here is all you need to know to solve this problem:

1. Because of angular momentum conservation, the motion lies in a plane.
2. Use polar coordinates r and θ to describe the motion, with the sun at the center of the coordinate system. (The usual rectangular coordinates are given by $x = r\cos\theta$ and $y = r\sin\theta$.)
3. The angular momentum
$$L = m\,r^2\,\dot\theta \tag{6}$$
is a constant of the motion. ($\dot\theta$ is classical mechanics shorthand for $\frac{dr}{dt}$.)
4. The kinetic energy is
$$KE = (1/2)\,m\,v^2 \tag{7}$$
where
$$v^2 = \dot r^2 + (r\dot\theta)^2\ . \tag{8}$$
5. The total energy
$$E = KE + V \tag{9}$$
is a constant of the motion.
6. The initial conditions can be described by r, $\dot r$, θ, and $\dot\theta$ at time $t = 0$. You can use the two constants of motion L and E to find the motion. (This is equivalent to $\vec F = m\,\vec a$, but much easier!)

Further hints:

(1) The above equations of motion involve $\dot r = \frac{dr}{dt}$ and $\dot\theta = \frac{d\theta}{dt}$. You can take the ratio of those: $\dot\theta/\dot r = \frac{d\theta}{dr}$, and thereby get $\frac{d\theta}{dr}$ as a function of r, which can be integrated to get θ as a function of r, which describes the orbit.

(2) The obvious constants of motion are the total energy E and the total angular momentum L as described above. But it is more convenient to use r_0 = distance of closest approach, and v_0 = velocity at the point of closest approach. It is easy to compute E and L in terms of those two quantities, since at the point of closest approach, $\dot r = 0$.

(3) It may help to choose explict values for r_0 and v_0 before asking Mathematica to do the required integral, because it may otherwise like to give you a result containing imaginary numbers.