Worksheet #9 – PHY102 (Spring 2011)

More on `Do` loops: Intentional chaos

Tools you will need

This week you will need to use `ListPlot`, `Animate`, and `Do` or `For`. You can review these in your notebook from Worksheet 6, where they were introduced; or you can look them up in the online help. You will also want to use `Table` or `NestList`, so have a look at those in the online help and make up some examples for yourself to get familiar with them before starting in on the assigned problem.

The new physics – Chaos

Chaos, though it had been discussed extensively for a couple of centuries (e.g. Boltzmann and Maxwell discussed "molecular chaos"), has really come into its own since the widespread use of computers. That is because the solutions to chaotic systems—even simple ones—do not lend themselves to the kind of mathematical closed-form solutions that are can be handled by traditional analytic mathematical methods.

An early surprise was that even quite simple looking systems can display chaotic behavior, whereas it was originally thought that chaos only occured in systems with billions of molecules. In this worksheet, you will study perhaps the simplest system which shows chaos: namely, the purely mathematical nonlinear "mapping"

$$x_{n+1} = \lambda\, x_n\,(1 - x_n) \tag{1}$$

This mapping model can be used, for example, to describe how a population density, x_n changes from one generation (n) to the next ($n+1$). Actually, it is not a very realistic model; but it does illustrate many of the features of more complex systems. The parameter λ can be considered to be the "birth rate", *i.e.*, the number of offspring from the last generation. The way it works is that if we know the population density at some time and call that density x_1, then the population density of the next generation is $x_2 = \lambda\, x_1\,(1 - x_1)$. This procedure is continued using Eq. 1 to find the population density for later generations. Intuitively, chaos means a lack of order. Mathematically, it is defined by how stable the behavior of a set of equations is to small perturbations in the initial conditions. In the context of equation 1, this means how stable is the series of iterates $(x_1, x_2, x_3, \dots)$ when you make a very small change $x_1 \rightarrow x_1^\delta = x_1 + \delta x$. When this change is made, we get a new set of iterates $(x_1^\delta, x_2^\delta, x_3^\delta, \dots)$.

If a set of equations is in a chaotic regime then the trajectories defined by these series of points diverge exponentially. In the context of our example,

$$|x_n^\delta - x_n| \sim e^{\nu n}, \tag{2}$$

where in a *chaotic* system, the *Lyapunov exponent* ν is positive.

Problem 1

(i) Write a Mathematica code to iterate the mapping in Eq. (1). (You can use `Do`, `For`, or `NestList` for it. Another useful function is `Range`, which can be used to specify the dimension of an array—even if you don't want to use the values that `Range` puts into that array.) Plot the steady-state behavior of the mapping as a function of the parameter λ for $1 < \lambda < 4$. Do this for several different values of the starting point x_1 in the range $0 < x_1 < 0.5$.

(ii) For some particular value of λ in the regime that looks chaotic in your graph, make an estimate of the Lyapunov exponent using Eq. 2.

Hint: choose x_1 and $x_1^\delta = x_1 + \delta x$, where δx is small. Plot the series $x_n^\delta - x_n$ as a function of n and look for exponential growth on the average. Since you are looking for $x_n^\delta - x_n \approx \text{const} \times e^{\nu n}$, you might want to also plot the sequence $\log(|x_n^\delta - x_n|)$ or even $(\frac{1}{n}) \log(|x_n^\delta - x_n|)$. If all is well, your estimate of the Lyapunov exponent ν will be independent of the choices of x_1 and δx.