Worksheet #10 – PHY102 (Spring 2011)

The wave equation and the diffusion equation

In this worksheet we study two *partial differential equations* that are very important in physics.

Many wave motions can be described by the linear wave equation. We shall do problems concerning waves on a string, but the equation we study has many other applications. For example atomic vibrations in solids, light waves, sound waves and water waves are all described by similar equations. The linear wave equation for waves on a string is the partial differential equation

$$\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2}\,, \tag{1}$$

where $y(x,t)$ is the distance by which the string is displaced at location x and time t, and v is the speed of the waves. For transverse waves on an actual string, the wave speed is related to the tension T and mass-per-unit-length μ of the string by $v = \sqrt{T/\mu}$ (see e.g., Halliday and Resnick for the derivation).

A second partial differential equation that is very important in physics is the diffusion equation. Atoms in a gas diffuse in a manner described by this equation. Similarly, pollutants in the ground often diffuse through the soil. This motion is very different from wave motion. In general each physical system has ranges of parameters where the motion is “diffusive” or “wavelike”. In solids, for example, motion is wavelike at short times and over long distances (e.g., sound waves), but diffusive on long times and short distances (atomic hops). The diffusion equation is given by,

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}\,. \tag{2}$$

Here $c(x,t)$ is the concentration of diffusing atoms at position x at time t. You can imagine putting a drop of ink in water and watching the color spread. In that case, $c(x,t)$ is the density of ink. D is the “diffusion constant” which sets the rate at which the spreading occurs.

Problem 1 – Wave Phenomena

(i) Standing waves. Consider waves on a string of length L. The transverse displacement at each end of the string is fixed at zero. Check that the two solutions: $y1(x,t) = y_m \sin(kx - \omega t)$, and $y2(x,t) = y_m \sin(kx + \omega t)$, both satisfy the wave equation, provided that k and ω are related in the appropriate way. Show that $y1(x,t) + y2(x,t)$ satisfies the boundary condition $y(0,t) = 0$ for all t. Find the condition required to also satisfy the boundary condition $y(L,t) = 0$ for all t (you can use units where $L = 1$.)

If we seek the “fundamental mode” (i.e., the lowest frequency mode of vibration), how are k and ω related to v and the length of the string? Set $k = k_0$ and $\omega = \omega_0$ (i.e., the values for the fundamental mode). Show that $y(x,t) = y1(x,t) + y2(x,t)$ gives rise to standing waves. Animate the solution $y1$ and the solution $y = y1 + y2$.

(ii) Beats. Now consider two solutions of the form $y1(x,t) = y_m \cos(kx - \omega_1 t)$ and $y2(x,t) = y_m \cos(kx - \omega_2 t)$, where $\omega_1 = \omega + \delta\omega$ and $\omega_2 = \omega - \delta\omega$ Check that the linear superposition of these two propagating waves produces a beat pattern. How does the beat frequency depend on $\delta\omega$?

(iii) Superposition. Almost all functions can be written as a superposition of sine and cosine waves. As an example, consider the linear superposition of sine waves such that;

$$y(x) = \sum_{n=1}^{n_{\max}} \frac{1}{n} \sin(2\,\pi\,n\,x/L) \tag{3}$$

Check the evolving pattern as $n_{\max}$ is increased. Make plots of $y(x)$ for $n_{\max} = 3, 10, 100$ terms. Can you identify limiting curve as $n_{\max}$ becomes large?

Problem 2 – Diffusion.

Show that $c(x,t) = \frac{1}{\sqrt{2\,D\,t}} \exp(-\frac{x^2}{4\,D\,t})$ satisfies the diffusion equation. Animate the plots of $c(x,t)$ for different values of t. Notice that the amplitude of $c(x,t)$ decays with time: this is the essence of "diffusion." In contrast, in the linear wave equation, the wave amplitude remained constant: it propagates instead of spreading.

(In reality there is usually some "damping" of waves. This is often modeled by adding a "diffusion term" to the wave equation—analogous to the damping term we sometimes add to the equation of motion of a harmonic oscillator.)