

# Worksheet #11 – PHY102 (Spring 2011)

## DC and AC circuits

In earlier worksheets we have studied the behavior of a damped mass+spring system. We also took a brief look at linear and non-linear pendulum problems. The equations describing these problems also describe the behavior of linear and non-linear electrical circuits, and there are many analogies between mechanical and electrical problems that are often used in physics discussions. For example the concepts of underdamped, critically damped and overdamped apply to electrical problems as well as to mechanical problems. Similarly the concept of resonance applies to both mechanical and electrical problems. In circuits, a resistor absorbs energy and acts as the damping. In mechanical problems, energy is transferred between potential energy and kinetic energy. In electrical problems, energy is transferred between energy stored in the magnetic field and energy stored in the electrical field. In the case of electrical circuits, energy is transferred between capacitors and inductors.

The relation between the magnitudes of the voltage (V) and current (I) in circuit elements L(=Inductance), R(=Resistance), and C(=Capacitance) are as follows:

$$\begin{aligned}V_R &= IR \\V_L &= L \frac{dI}{dt} \\V_C &= \frac{1}{C} \int I dt .\end{aligned}$$

Kirchhoff's voltage law states that the sum of the voltage drops around a series LRC circuit must sum to zero, yielding the following equation,

$$V_s - V_R - V_L - V_C = 0 \tag{1}$$

where  $V_s$  represents the voltage supplied by a source such as a battery or electrical generator. Using the relations above between the current and voltages for L,R,C given above we find,

$$V_s - IR - L \frac{dI}{dt} - \frac{1}{C} \int I dt = 0 \tag{2}$$

Taking the derivative of this equation with respect to time yields,

$$\frac{dV_s}{dt} - L \frac{d^2I}{dt^2} - R \frac{dI}{dt} - \frac{I}{C} = 0 \tag{3}$$

Recall that a damped mass-spring system is described by  $F_s - bv - kx = ma$ , so that,

$$F_s - m \frac{d^2x}{dt^2} - b \frac{dx}{dt} - kx = 0 \tag{4}$$

where  $F_s$  is the driving force on the mass-spring system. The analogy between the mass+spring system and the electric circuit are evident from these equations:  $L$  acts like the mass,  $R$  like the

damping and  $1/C$  like the spring constant. The derivative of the source voltage acts like the driving force.

### Problem 1 – Transients

Consider a series LRC circuit connected to DC voltage of 1 Volt at time 0. More specifically a resistor with resistance  $R$ , an initially-uncharged capacitor with capacitance  $C$ , an inductor with inductance  $L$ , an ideal battery of with voltage 1V, and a switch are connected in series. The switch is open for  $t < 0$  and closed for  $t > 0$ .) Write a Mathematica code to find and plot the current in the LRC circuit as a function of time. Run your code for various values of  $L$ ,  $R$  and  $C$  to illustrate the cases of underdamped and overdamped circuits.

Hint: in the underdamped case, Mathematica may give you the solution in a form that contains complex exponentials instead of a nicer equivalent form of trig function times exponential with no imaginary terms. To convert it, you may need to use

```
FullSimplify[ExpToTrig[ ... ]]
```

### Problem 2 – Resonance

Consider an LRC circuit driven by a steady sinusoidal AC source with rms amplitude 1 Volt. Find and plot the amplitude of the steady state current in the circuit as a function of the frequency  $\omega$  of the AC source, which supplies a voltage

$$V_s = V_{max} \cos[\omega t + \phi]$$

You can take the phase  $\phi$  of the applied voltage to be zero. The phase of the current will then *not* be zero.

Note that the frequencies used here are the “angular frequency” measured in radians per second, as opposed to  $f = \omega/(2\pi)$ , which is measured in cycles per second (Hz.)

Hint # 1: In principle, you can solve this using DSolve. That is somewhat tricky, because you have to get rid of the transient terms in the solution, which decay exponentially. An alternative is to just assume that the answer will have a current of the form  $i = A \cos \omega t + \theta$  and find the value for the amplitude  $A$  and phase  $\theta$  that satisfies Eq. (1).

Hint #2: You will need to get Mathematica to transform something of the form  $c_1 \cos(\omega t) + c_2 \sin(\omega t)$  into the form  $c_3 \cos(\omega t + c_4)$ . To help with that, see what happens when you run

```
y = Cos[omega*t + phi]
```

```
z = TrigExpand[y]
```

```
Coefficient[z,Cos[omega t]]
```

Also plot the voltage across the capacitor and the voltage across the inductor on the same graph. Make separate graphs for situations where the applied frequency is *above*, *below*, and *at* the resonant frequency. Choose the other parameters such that the damping is significant but not too strong.