

$$1.(a) T = \frac{1}{2} M v^2$$

$$= \frac{1}{2} M [(R\dot{\theta})^2 + (R \sin\theta \omega)^2]$$

$$V = \text{"mgh"} = -mgR \cos\theta$$

$$\mathcal{L} = T - V = \frac{1}{2} MR^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta) + MgR \cos\theta$$

$$(b) \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\frac{d}{dt} [MR^2 \dot{\theta}] = \frac{1}{2} MR^2 (2\omega^2 \sin\theta \cos\theta) - MgR \sin\theta$$

$$MR^2 \ddot{\theta} = MR^2 \omega^2 \sin\theta \cos\theta - MgR \sin\theta$$

$$\ddot{\theta} = \omega^2 \sin\theta \cos\theta - \frac{g}{R} \sin\theta$$

(c)  $T+V$  is not constant for this problem, because the torque that makes  $\omega$  be constant depends on  $\theta$  and  $\dot{\theta}$ .

Specifically,

$$T+V = \frac{1}{2}MR^2(\dot{\theta}^2 + \omega^2 \sin^2 \theta) - MgR \cos \theta$$

$$\begin{aligned} \frac{d}{dt}(T+V) &= \frac{1}{2}MR^2(2\dot{\theta}\ddot{\theta} + 2\omega^2 \sin \theta \cos \theta \dot{\theta}) \\ &\quad + MgR \sin \theta \dot{\theta} \\ &= \dot{\theta} [MR^2\ddot{\theta} + MR^2\omega^2 \sin \theta \cos \theta \\ &\quad + MgR \sin \theta] \\ &= \dot{\theta} [MR^2] \left[ \omega^2 \sin \theta \cos \theta - \frac{g}{R} \sin \theta \right. \\ &\quad \left. + \omega^2 \sin \theta \cos \theta + \frac{g}{R} \sin \theta \right] \end{aligned}$$

$$\frac{d}{dt}(T+V) = 2MR^2\omega^2 \sin\alpha \cos\alpha \dot{\alpha}$$

this is not 0 (unless  $\omega = 0$  or  $\alpha = 0$ )

$$\begin{aligned} 2.0(a) T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2} m \left[ (\dot{\rho} \cos\phi - \rho \sin\phi \dot{\phi})^2 \right. \\ &\quad \left. + (\dot{\rho} \sin\phi + \rho \cos\phi \dot{\phi})^2 \right. \\ &\quad \left. + (\dot{\rho} / \tan\alpha)^2 \right] \\ &= \frac{1}{2} m \left[ \dot{\rho}^2 + (\rho \dot{\phi})^2 + \dot{\rho}^2 \cos^2\alpha / \sin^2\alpha \right] \\ &= \frac{1}{2} m \left[ \dot{\rho}^2 / \sin^2\alpha + (\rho \dot{\phi})^2 \right] \end{aligned}$$

$$V = mgz = mg\rho / \tan\alpha$$

$$\mathcal{L} = T - V = \frac{1}{2} m \left[ \dot{\rho}^2 / \sin^2\alpha + (\rho \dot{\phi})^2 \right] - mg\rho / \tan\alpha$$

$$(b) \quad \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\rho}} \right) = \frac{\partial \mathcal{L}}{\partial \rho}$$

$$\Rightarrow \frac{d}{dt} \left( m\dot{\rho} / \sin^2\alpha \right) = m\rho \dot{\phi}^2 - mg / \tan\alpha$$

$$\ddot{\rho} / \sin^2 \alpha = \rho \dot{\phi}^2 - g / \tan \alpha$$

and  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$

$$\Rightarrow \frac{d}{dt} [m \rho^2 \dot{\phi}] = 0$$

$$\Rightarrow (\rho^2 \dot{\phi}) = \text{constant}$$

Remarks — not required —

(1)  $\rho^2 \dot{\phi} = \text{constant}$  is angular momentum constant (no torque)

(2)  $T + V = \text{constant}$ , so

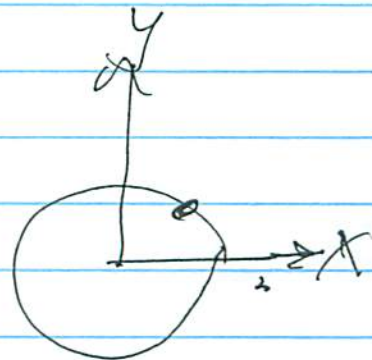
$$E = \frac{1}{2} m \left[ \dot{\rho}^2 / \sin^2 \alpha + \rho^2 \dot{\phi}^2 \right] + m g \rho / \tan \alpha$$

$$= \frac{1}{2} m \left[ \dot{\rho}^2 / \sin^2 \alpha + \frac{1}{\rho^2} (\underbrace{\rho^2 \dot{\phi}}_{\text{const.}})^2 \right] + m g \rho / \tan \alpha$$

3. Point P is at

$$x = R \cos(\omega t)$$

$$y = R \sin(\omega t)$$



Center of mass is at

$$\begin{pmatrix} x_{cm} \\ y_{cm} \end{pmatrix} = \begin{pmatrix} R \cos(\omega t) \\ R \sin(\omega t) \end{pmatrix} + \begin{pmatrix} \frac{l}{2} \sin \phi \\ -\frac{l}{2} \cos \phi \end{pmatrix}$$

$$T_{cm} = \frac{1}{2} m (\dot{x}_{cm}^2 + \dot{y}_{cm}^2)$$

$$= \frac{1}{2} m \left\{ \left( -R\omega \sin \omega t + \frac{l}{2} \cos \phi \dot{\phi} \right)^2 + \left( R\omega \cos \omega t + \frac{l}{2} \sin \phi \dot{\phi} \right)^2 \right\}$$

$$= \frac{1}{2} m \left[ (R\omega)^2 + \left( \frac{l}{2} \dot{\phi} \right)^2 \right]$$

$$+ R\omega l \dot{\phi} \left( -\cos \phi \sin \omega t + \sin \phi \cos \omega t \right)$$

Also have

$$T_{rot} = \frac{1}{2} \left( \frac{Ml^2}{12} \right) \dot{\phi}^2$$

## HW 11.6

$$\begin{aligned} T &= T_{\text{tan}} + T_{\text{rot}} \\ &= \frac{1}{2} M \left[ (R\omega)^2 + \left( \frac{l}{2} \dot{\phi} \right)^2 \right. \\ &\quad \left. + R\omega l \dot{\phi} (-\cos\phi \sin\omega t + \sin\phi \cos\omega t) \right. \\ &\quad \left. + \frac{l^2}{12} \dot{\phi}^2 \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} M \left[ (R\omega)^2 + \frac{l^2 \dot{\phi}^2}{3} \right. \\ &\quad \left. + R\omega l \dot{\phi} (-\cos\phi \sin\omega t + \sin\phi \cos\omega t) \right] \end{aligned}$$

$$V = Mg y_{\text{cm}} = Mg \left( R \sin\omega t - \frac{l}{2} \cos\phi \right)$$

$$\mathcal{L} = T - V$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} M \left[ (R\omega)^2 + \frac{l^2 \dot{\phi}^2}{3} \right. \\ &\quad \left. + R\omega l \dot{\phi} (\sin\phi \cos\omega t - \cos\phi \sin\omega t) \right] \\ &\quad - Mg \left( R \sin\omega t - \frac{l}{2} \cos\phi \right) \end{aligned}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

$$\frac{d}{dt} \left\{ \frac{M l^2 \dot{\phi}}{3} \right.$$

$$\left. + \frac{1}{2} M R \omega l (\sin \phi \omega \omega t - \cos \phi \sin \omega t) \right\}$$

$$= \frac{1}{2} M R \omega l \dot{\phi} (\cos \phi \omega \omega t + \sin \phi \sin \omega t)$$

$$+ \frac{M g l}{2} (-\sin \phi)$$

$$\frac{M l^2}{3} \ddot{\phi} + \frac{1}{2} M R \omega l (-\sin \phi \sin \omega t \omega$$

$$+ \underbrace{\cos \phi \dot{\phi} \omega \omega t}$$

$$- \cos \phi \cos(\omega t) \omega$$

$$+ \underbrace{\sin \phi \dot{\phi} \sin \omega t})$$

$$= \frac{1}{2} M R \omega l \dot{\phi} (\underbrace{\cos \phi \omega \omega t + \sin \phi \sin \omega t})$$

$$- \frac{M g l}{2} \sin \phi$$

divide by  $M$

$$\frac{l^2}{3} \ddot{\phi} - \frac{1}{2} R \omega^2 l (\sin \phi \sin \omega t + \cos \phi \cos \omega t)$$

$$+ \frac{g l}{2} \sin \phi = 0$$

$$\ddot{\phi} - \frac{3}{2} \frac{R \omega^2}{l} (\sin \phi \sin \omega t + \cos \phi \cos \omega t)$$

$$+ \frac{3}{2} \frac{g}{l} \sin \phi = 0$$

or

$$\ddot{\phi} - \frac{3}{2l} R \omega^2 \cos(\phi - \omega t)$$

$$+ \frac{3}{2} \frac{g}{l} \sin \phi = 0$$



$$4. (a) \begin{cases} x = R \sin \theta \cos \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \theta \end{cases}$$

$$\dot{x} = R \sin \theta (-\sin \phi \dot{\phi}) + R \cos \theta \cos \phi \dot{\theta}$$

$$\dot{y} = R \sin \theta (\cos \phi \dot{\phi}) + R \cos \theta \sin \phi \dot{\theta}$$

$$\dot{z} = -R \sin \theta \dot{\theta}$$

$$T = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2} M R^2 \left[ (-\sin \theta \sin \phi \dot{\phi} + \cos \theta \cos \phi \dot{\theta})^2 + (\sin \theta \cos \phi \dot{\phi} + \cos \theta \sin \phi \dot{\theta})^2 + (\sin \theta \dot{\theta})^2 \right]$$

$$= \frac{1}{2} M R^2 \left( \sin^2 \theta \dot{\phi}^2 + \cos^2 \theta \dot{\theta}^2 + \sin^2 \theta \dot{\theta}^2 \right)$$

$$= \frac{1}{2} M R^2 \left[ \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right]$$

$$V = -MgZ = -MgR \cos \theta$$

↑  
Because Z axis points  
down

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} MR^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

$$+ MgR \cos \theta$$

$$p_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = MR^2 \sin^2 \theta \dot{\phi}$$

$$F_{\phi} = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$p_{\phi} = \bar{F}_{\phi} \Rightarrow \boxed{MR^2 \sin^2 \theta \dot{\phi} = \text{const}}$$

$$\text{or } \sin^2 \theta \ddot{\phi} + 2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} = 0$$

$$p_{\theta} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = MR^2 \dot{\theta}$$

$$F_{\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = MR^2 \sin \theta \cos \theta \dot{\phi}^2 - MgR \sin \theta$$

$$P_\theta = F_\theta \Rightarrow$$

$$\begin{aligned} MR^2 \ddot{\theta} &= MR^2 \sin\theta \cos\theta \dot{\phi}^2 \\ &\quad - MgR \sin\theta \end{aligned}$$

4(b) No  $\phi$  in  $L$ , so

$$\begin{aligned} p_\phi &= z\text{-comp. of angular momentum} \\ &= \text{constant} \end{aligned}$$