Phy 321 Spring 2017
HW 12.1

1. (a) $T = \frac{1}{2} m\left(\dot{r}^2 + (r\dot{\theta})^2\right)$

$$V = -mgr\cos\theta + \frac{1}{2}k(r-B)^2$$

$$\mathcal{L} = T - V$$

$$\boxed{\mathcal{L} = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}k(r-B)^2}$$

(b) $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$

$$F_r = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - k(r-B)$$

$$\dot{p}_r = F_r \Rightarrow \boxed{m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - k(r-B)}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mgr\sin\theta$$

$$\dot{p}_\theta = F_\theta \Rightarrow \boxed{mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta}$$

HW12.2

Divide $\dot{p}_r = F_r$ by M =

$$\boxed{\ddot{r} - r\dot{\theta}^2 - g\cos\theta + \frac{k}{M}(r-B) = 0}$$

Divide $\dot{p}_\theta = F_\theta$ by Mr

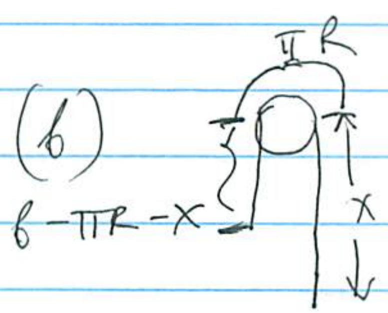
$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0}$$

2. (a) $T = \frac{1}{2}(b\sigma)\dot{x}^2 + \frac{1}{2}I\omega^2$

where $\omega = v/R = \dot{x}/R$

$$\Rightarrow T = \frac{1}{2}\left(b\sigma + \frac{I}{R^2}\right)\dot{x}^2$$

(b)



$$V = -(\sigma x)\frac{x}{2} - \frac{(\sigma(b - \pi R - x))(b - \pi R - x)}{2}$$

$$= -\frac{\sigma x^2}{2} - \frac{\sigma}{2}(b - \pi R - x)^2$$

$$= -\sigma x^2 + \sigma x(b - \pi R) + \text{const}$$

$$= -\sigma\left(x - \frac{b - \pi R}{2}\right)^2 + \text{const}$$

2.(c) $L = T - V$

$$= \frac{1}{2}\left(b\sigma + \frac{I}{R^2}\right)\dot{x}^2$$

$$+ \sigma\left(x - \frac{b - \pi R}{2}\right)^2$$

$$+ \underbrace{\text{const}}_{\text{(don't care)}}$$

$$p = \frac{\partial L}{\partial \dot{x}} = \left(b\sigma + \frac{I}{R^2}\right)\dot{x}$$

$$F = \frac{\partial L}{\partial x} = 2\sigma\left(x - \frac{b - \pi R}{2}\right)$$

$F = \dot{p} \Rightarrow$

$$\left(b\sigma + \frac{I}{R^2}\right)\ddot{x} = 2\sigma\left(x - \frac{b - \pi R}{2}\right)$$

2.(d)

$$E = T + V = \frac{1}{2}\left(b\sigma + \frac{I}{R^2}\right)\dot{x}^2$$

$$- \sigma\left(x - \frac{b - \pi R}{2}\right)^2$$

$$0 = \frac{dE}{dt} = \left(b\sigma + \frac{I}{R^2}\right)\dot{x}\ddot{x}$$

$$- 2\sigma\left(x - \frac{b - \pi R}{2}\right)\dot{x}$$

$$\Rightarrow \left(b\sigma + \frac{I}{R^2}\right)\ddot{x} - 2\sigma\left(x - \frac{b - \pi R}{2}\right) = 0 \checkmark$$

2. (e) $\left(b\sigma + \frac{I}{R^2}\right)\ddot{x} - 2\sigma\left(x - \frac{b-\pi R}{2}\right) = 0$

Let $x - \frac{b-\pi R}{2} = y$

$$\left(b\sigma + \frac{I}{R^2}\right)\ddot{y} - 2\sigma y = 0$$

$$y = C_1 \sinh\left(\sqrt{\frac{2\sigma}{b\sigma + \frac{I}{R^2}}}\, t\right) + C_2 \cosh\left(\sqrt{\frac{2\sigma}{b\sigma + \frac{I}{R^2}}}\, t\right)$$

$$x = \frac{b-\pi R}{2} + C_1 \sinh\left(\sqrt{\frac{2\sigma}{b\sigma + \frac{I}{R^2}}}\, t\right) + C_2 \cosh\left(\sqrt{\frac{2\sigma}{b\sigma + \frac{I}{R^2}}}\, t\right)$$

$\dot{x} = 0$ at $t = 0 \Rightarrow C_1 = 0$

$x = x_0$ at $t = 0 \Rightarrow$

$$x_0 = \frac{b-\pi R}{2} + C_2$$

HW 12.5

2. (e)

$$x = \frac{b - \pi R}{2} + \left(x_0 - \frac{b-\pi R}{2}\right)\cosh\left[\sqrt{\frac{2\sigma}{b\sigma + \frac{I}{R^2}}}\, t\right]$$

3. (a)

$$\begin{cases} x_{cm} = \frac{l}{2}\sin\phi \\ y_{cm} = R\sin\omega t + \frac{l}{2}\cos\phi \end{cases}$$

$$\begin{cases} \dot{x}_{cm} = \frac{l}{2}\cos\phi\,\dot{\phi} \\ \dot{y}_{cm} = \omega R\cos\omega t - \frac{l}{2}\sin\phi\,\dot{\phi} \end{cases}$$

$$T = \frac{1}{2}M\left[\left(\frac{l}{2}\cos\phi\,\dot{\phi}\right)^2 + \left(\omega R\cos\omega t - \frac{l}{2}\sin\phi\,\dot{\phi}\right)^2\right] + \frac{1}{2}\frac{Ml^2}{12}\dot{\phi}^2$$

$$= \frac{1}{2}M\left[\frac{l^2}{4}\dot{\phi}^2 + (\omega R\cos\omega t)^2 - \omega Rl\cos\omega t\sin\phi\,\dot{\phi} + \frac{l^2}{12}\dot{\phi}^2\right]$$

HW 12.6

$$T = \frac{1}{2} M \left[\frac{l^2}{3} \dot{\phi}^2 - \omega R l \cos\omega t \sin\phi \, \dot{\phi} + (\omega R \cos \omega t)^2 \right]$$

$$V = M g \, y_{cm} = M g \left[R \sin\omega t + \frac{l}{2} \cos\phi \right]$$

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} M \left[\frac{l^2}{3} \dot{\phi}^2 - \omega R l \cos\omega t \sin\phi \, \dot{\phi} + (\omega R \cos\omega t)^2 \right] - M g \left[R \sin\omega t + \frac{l}{2} \cos\phi \right]$$

3. (b) $$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \cancel{= \frac{Ml^2}{3}\dot{\phi} + \frac{M\omega R l}{2}\cos}$$

$$= \frac{M l^2}{3} \dot{\phi} - \frac{M \omega R l}{2} \cos\omega t \sin\phi$$

$$F = \frac{\partial \mathcal{L}}{\partial \phi} = \frac{-M}{2} \omega R l \cos\omega t \cos\phi \, \dot{\phi} + M g \frac{l}{2} \sin\phi$$

HW 12.7

$\dot{p} = F \Rightarrow$

$$\frac{M\ell^2}{3}\ddot{\phi} + \frac{M\omega^2 R\ell}{2}\sin(\omega t)\sin\phi$$

$$\underbrace{- \frac{M\omega R\ell}{2}\cos(\omega t)\cos\phi\,\dot{\phi}}$$

$$= \underbrace{\frac{-M}{2}\omega R\ell\cos(\omega t)\cos\phi\,\dot{\phi}}$$

$$+ \frac{Mg\ell}{2}\sin\phi$$

$$\boxed{\frac{M\ell^2}{3}\ddot{\phi} + \frac{M\omega^2 R\ell}{2}\sin(\omega t)\sin\phi - \frac{Mg\ell}{2}\sin\phi = 0}$$

or can multiply by $\frac{3}{M\ell^2}$ to get

$$\boxed{\ddot{\phi} + \frac{3}{2}\frac{\omega^2 R}{\ell}\sin(\omega t)\sin\phi - \frac{3}{2}\frac{g}{\ell}\sin\phi = 0}$$

3. (c) with $g=1$, $l=1$, $h=0.05$ this becomes

$$\ddot{\phi} - \frac{3}{2}\sin(\phi) + 0.075\,\omega^2 \sin(\omega t)\sin(\phi) = 0$$

In Mathematica, let

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zero = phi''[t] - 3/2 sin[phi[t]]
      + 0.075 omega^2 sin[omega*t] sin[phi[t]]
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sol = NDSolve[{(zero /. omega -> 40) == 0,
               phi[0] == 0.4, phi'[0] == 0},
               phi[t], {t, 0, 20}]
Plot[phi[t] /. sol[[1]], {t, 0, 20}]
```

Find stable motion (with $-1.265 < \phi < 1.265$) if $\omega > 38.2361$

UNSTABLE if $\omega < 38.2360$

HW 12, 9

If change initial phi[0] from 0.4 to 0.3, then stable for

$\omega > 31.623$

with $-1.07 < \phi < 1.07$

If change initial phi[0] to 0.2, stable for $\omega > 27.68$

with $-0.846 < \phi < 0.846$