Phy 321 Spring 2017
HW 13.1

1. (a) $U = -\int^r dr' \, F(r')$

$= \int^r dr' \, \frac{A}{r'^3} = \left[\frac{A r'^{-2}}{-2} \right]^r$

$= \frac{-A}{2r^2} + \cancel{\text{const}}$

check this: $F = -\frac{dU}{dr} = (-1)\left(\frac{-A}{2}\right)\frac{-2}{r^3}$

$= \frac{-A}{r^3}$ ✓

(b) use polar coords

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$

$V = \frac{-A}{2r^2}$

$\mathcal{L} = T - V$

$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi}$

$F_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0$

$\dot{p}_\phi = F_\phi \Rightarrow p_\phi = \text{const} = \text{angular mom}$

HW 13.2

$$E = T + V$$

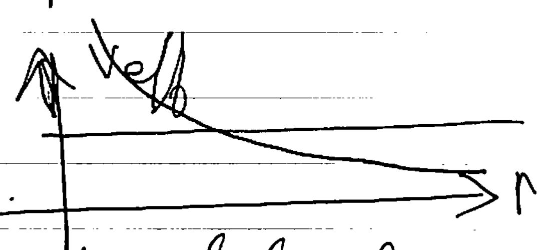
$$= \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{A}{2r^2}$$

$$= \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu \frac{1}{r^2} \left(\frac{p_\phi}{\mu}\right)^2 - \frac{A}{2r^2}$$

$$\boxed{\begin{aligned} E &= \frac{1}{2} \mu \dot{r}^2 + V_{eff}(r) \\ \text{where } V_{eff}(r) &= \frac{p_\phi^2}{2\mu r^2} - \frac{A}{2r^2} \end{aligned}}$$

1. (c) let $p_\phi = l$

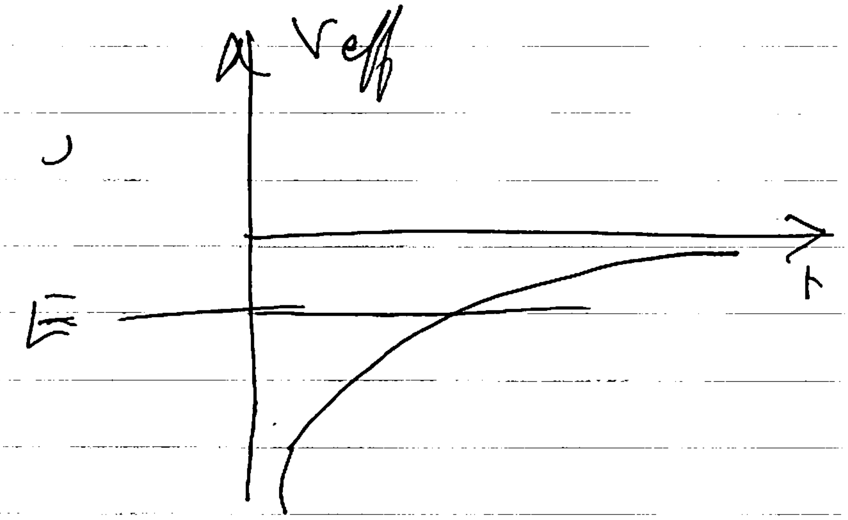
$$V_{eff} = \frac{1}{2}\left(\frac{l^2}{\mu} - A\right) \frac{1}{r^2}$$



If $\left(\frac{l^2}{\mu} - A\right) > 0$, ~~$V_{eff}$~~ orbit is unbounded always

HW 13, 3

If $\left(\frac{l^2}{M} - A\right) < 0$,



can have bounded orbit
(will be bounded whenever
$\frac{l^2}{M} - A < 0$ and $E < 0$

$\boxed{|l| < AM}$

1. (d) $l = p_\phi = M v_0 r_0$

$$V_{eff} = \frac{(M v_0 r_0)^2}{2 M r^2} - \frac{A}{2 r^2}$$

$$= \left(M v_0^2 r_0^2 - A\right) \frac{1}{2 r^2}$$

$$E = V_{eff}(r_0) = \frac{1}{2}\left(M v_0^2 - \frac{A}{r_0^2}\right)$$

because $\dot{r} = 0$ at $r = r_0$ from $t = 0$

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff} \Rightarrow$$

$$\frac{1}{2}\left(\mu v_0^2 - \frac{A}{r_0^2}\right) = \frac{1}{2}\mu \dot{r}^2 + \frac{(\mu v_0^2 r_0^2 - A)}{2r^2}$$

mult. by $\frac{2}{\mu}$

$$\left(v_0^2 - \frac{A}{\mu r_0^2}\right) - \frac{\left(v_0^2 r_0^2 - \frac{A}{\mu}\right)}{r^2} = \dot{r}^2$$

let $b = \sqrt{\frac{A}{\mu r_0^2} - v_0^2}$

$$-b^2 + \frac{r_0^2}{r^2} b^2 = \dot{r}^2$$

$$\frac{dr}{dt} = \pm\sqrt{-b^2\left(1 - \frac{r_0^2}{r^2}\right)} = \pm\sqrt{b^2\left(\frac{r_0^2}{r^2} - 1\right)}$$

$$\int \frac{dr}{\sqrt{\frac{r_0^2}{r^2} - 1}} = \int \pm b\, dt$$

$$\int \frac{r\,dr}{\sqrt{r_0^2 - r^2}} = \pm b t + \text{const}$$

let $\sqrt{r_0^2 - r^2} = u$

$$-r\,dr = u\,du$$

$$\int \frac{-u\,du}{u} = \pm b t + \text{const}$$

$$\sqrt{r_0^2 - r^2} = \pm b(t - t_0)$$

$$r_0^2 - r^2 = b^2 (t - t_0)^2$$

$$r = \sqrt{r_0^2 - b^2 (t - t_0)^2}$$

$\dot{r} = 0$ at $t = 0 \Rightarrow$

$$\boxed{r = \sqrt{r_0^2 - b^2 t^2} \quad \text{where } b = \sqrt{\frac{A}{\mu r_0^2} - v_0^2}}$$

1(e) $\quad l = P_\phi = \mu v_0 r_0 = \mu r^2 \dot{\phi}$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{v_0 r_0}{r^2} = \frac{v_0 r_0}{r_0^2 - b^2 t^2}$$

$$d\phi = \int \frac{v_0 r_0 \, dt}{r_0^2 - b^2 t^2} + \text{const}$$

$$= v_0 r_0 \int dt \, \frac{1}{2 r_0} \left[\frac{1}{r_0 + bt} + \frac{1}{r_0 - bt} \right]$$

$$\phi = \frac{v_0}{b} \operatorname{arctanh}\left(\frac{bt}{r_0}\right) + \text{const}$$

initial condition $\phi = 0$ at $t = 0 \Rightarrow$
const $= 0$

$$\boxed{\phi = \frac{v_0}{b} \operatorname{arctanh}\left(\frac{bt}{r_0}\right)}$$

can also write as $\phi = \frac{v_0}{2b} \log\left(\frac{r_0 + bt}{r_0 - bt}\right)$

remark (not required) motion ends when r hits 0, at time $t = \frac{r_0}{b}$
the velocity in radial direction $\dot{r}$ goes to infinity in that limit.

2. (a) $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$

$$V = -\frac{G m M}{r^2}$$

$$\mathcal{L} = T - V$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$F_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow p_\phi = \text{const}$$

HW 13.7

$E = T + V$ is constant

$$E = \frac{1}{2} m(\dot{r}^2) + \underbrace{\frac{p_\phi^2}{2mr^2} - \frac{GmM}{r}}_{V_{\text{eff}}(r)}$$

In the circular orbit,

$$p_\phi = mr^2\dot{\phi} = mR(R\dot{\phi}) = mvR$$

$$\frac{dV_{\text{eff}}}{dr} = 0 \Rightarrow \frac{p_\phi^2}{2m}\frac{-2}{r^3} - GmM\frac{-1}{r^2} = 0$$

$$\frac{p_\phi^2}{mR^3} = \frac{GmM}{R^2}$$

$$p_\phi^2 = (mvR)^2 = \frac{mR^3 GmM}{R^2}$$

$$v^2R^2 = RGM$$

$$\boxed{v = \sqrt{\frac{MG}{R}}}$$

another way to get it: $\frac{mv^2}{R} = F = \frac{GmM}{R^2}$

$$\Rightarrow v = \sqrt{\frac{MG}{R}} \checkmark$$

2.(b) After the burn (i.e. elliptical orbit)

$$E = \frac{1}{2} m \dot{r}^2 + \frac{p_\phi^2}{2mr^2} - \frac{GmM}{r}$$

turning points, where $\dot{r} = 0$, are at

$r = R$ and $r = 3R$

$$\begin{cases} E = \dfrac{p_\phi^2}{2mR^2} - \dfrac{GmM}{R} \\ E = \dfrac{p_\phi^2}{2m(3R)^2} - \dfrac{GmM}{(3R)} \end{cases}$$

Subtract these to cancel E

$$0 = \frac{p_\phi^2}{2m}\left(\frac{1}{R^2} - \frac{1}{9R^2}\right) - GmM\left(\frac{1}{R} - \frac{1}{3R}\right)$$

$$\frac{p_\phi^2}{2mR^2}\left(\frac{8}{9}\right) = \frac{GmM}{R}\left(\frac{2}{3}\right)$$

$$p_\phi^2 = \underline{\left(2Gm^2MR\right)}\,\frac{3}{4}$$

also $p_\phi = m v_b R$ where v_b = velocity just after burn

HW 13.9

$$(m v_b R)^2 = \frac{3}{2} G m^2 M R$$

$$\boxed{v_b = \sqrt{\frac{3}{2} \frac{GM}{R}}}$$

2. (c) $\ell_\phi = m v_b R$ just after burn

ℓ_ϕ is conserved, so $\ell_\phi = m v_A (3R)$ where v_A = velocity at apogee

$$m v_A (3R) = m v_b R$$

$$\boxed{v_A = \frac{1}{3} v_b = \sqrt{\frac{1}{6} \frac{GM}{R}}}$$