Phy321 Spring 2017
HW7.1

1. Use Energy conservation:

$$KE = \text{"}\frac{1}{2} M v^2\text{"} = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} M \left(\frac{H}{2}\right)^2 \left( \left[ \left(\frac{1}{2} + \cos u\right) \dot{u} \right]^2 + \left[ (\sin u)\, \dot{u} \right]^2 \right)$$

$$= \frac{1}{2} M \frac{H^2}{4} \left[ \left(\frac{1}{2} + \cos u\right)^2 + (\sin u)^2 \right] \dot{u}^2$$

$$= \frac{MH^2}{8} \left[ \frac{1}{4} + 1 + \cos u \right] \dot{u}^2$$

$$= \frac{MH^2}{8} \left( \frac{5}{4} + \cos u \right) \dot{u}^2$$

$$PE = \text{"}Mgh\text{"} = Mgy = Mg\frac{H}{2}(1 - \cos u)$$

$$KE + PE = \text{const} = \frac{1}{2} M v_0^2 + Mg\underbrace{y}_{0} = \frac{1}{2} M v_0^2$$

at $u = 0$

$$\therefore \quad \frac{MH^2}{8}\left(\frac{5}{4} + \cos u\right)\dot{u}^2 + \frac{MgH}{2}(1 - \cos u) = \frac{1}{2} M v_0^2$$

[Eq. 1]

Can also take $\frac{d}{dt}$ of the Energy Equation:

$$\frac{MH^2}{8}\left[\left(\frac{5}{4}+\cos u\right)2\dot{u}\ddot{u} - \sin u\,\dot{u}^3\right] + \frac{MgH}{2}\sin u\,\dot{u} = 0$$

divide by $\dot{u}$ $\Rightarrow$

$$\frac{MH^2}{8}\left[\left(\frac{5}{4}+\cos u\right)2\ddot{u} - \sin u\,\dot{u}^2\right] + \frac{MgH}{2}\sin u = 0 \qquad (\text{Eq. } 2)$$

$$F_x = M\ddot{x}$$

$$= M\frac{d}{dt}\frac{d}{dt}\left[\frac{H}{2}\left(\frac{u}{2}+\sin u\right)\right]$$

$$= \frac{MH}{2}\frac{d}{dt}\left(\left[\frac{1}{2}+\cos u\right]\dot{u}\right)$$

$$= \frac{MH}{2}\left(\left[\frac{1}{2}+\cos u\right]\ddot{u} - \sin u\,\dot{u}^2\right)$$

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$$F_y = M\ddot{y} = M\frac{d}{dt}\frac{d}{dt}\left[\frac{H}{2}(1-\cos u)\right]$$

$$= \frac{MH}{2}\frac{d}{dt}(\sin u\ \dot{u})$$

$$= \frac{MH}{2}\left(\sin u\ \ddot{u} + \cos u\ \dot{u}^2\right)$$

At $u = \pi$, the Energy conservation Eq. gives

$$\frac{MH^2}{8}\left(\frac{5}{4}-1\right)\dot{u}^2 + \frac{MgH}{2}(2) = \frac{1}{2}Mv_0^2$$

$$\frac{MH^2}{32}\dot{u}^2 + MgH = \frac{1}{2}Mv_0^2$$

$$\Rightarrow \dot{u}^2 = \frac{32}{MH^2}\left(-MgH + \frac{1}{2}Mv_0^2\right)$$

$$= \frac{32}{H^2}\left(-gH + \frac{1}{2}v_0^2\right)$$

the derivative of Energy Cons. Eq. gives

$$\frac{MH^2}{8}\left[\left(\frac{5}{4}-1\right)2\ddot{u}\right] = 0$$

$$\Rightarrow \ddot{u} = 0$$

Hence at $u = \pi$,

$$F_x = \frac{MH}{2}\left(\left(\frac{1}{2}-1\right)\cdot 0 - 0\cdot \dot{u}^2\right) = 0$$

$$F_y = \left(\frac{MH}{2}\right)\left(\cos(\pi)\,\dot{u}^2\right)$$

$$= \left(\frac{MH}{2}\right)(-1)\,\frac{32}{H^2}\left(-gH + \frac{1}{2}v_0^2\right)$$

$$= 16Mg - \frac{8Mv_0^2}{H}$$

$F_y$ is the total force in $y$ dir

$$F_y = F_y^{Track} - Mg$$

$$\boxed{\begin{aligned} F_x^{track} &= 0 \\ F_y^{track} &= 17Mg - \frac{8Mv_0^2}{H} \end{aligned}}$$

2. $$\frac{dx}{dt} = \frac{2x+1}{t+2}$$

$$\int \frac{dx}{2x+1} = \int \frac{dt}{t+2}$$

$$\frac{1}{2}\log\left(x+\frac{1}{2}\right) = \log(t+2) + \text{const}$$

$$\log\left(x+\frac{1}{2}\right) = 2\log(t+2) + \text{const}'$$

$$\log\frac{x+\frac{1}{2}}{(t+2)^2} = \text{const}'$$

$$x+\frac{1}{2} = (t+2)^2 * \text{const}$$

$$x = -\frac{1}{2} + C(t+2)^2$$

at $t=0$: $\quad 0 = -\frac{1}{2} + C\cdot 4$

$$\Rightarrow C = \frac{1}{8}$$

$$\boxed{x = -\frac{1}{2} + \frac{1}{8}(t+2)^2}$$

or $\boxed{x = \frac{1}{8}t^2 + \frac{1}{2}t}$ or $x = \frac{t(t+4)}{8}$

3. (a) $\ddot{x} + Ax = B$

$$x = C_1 \cos(\sqrt{A}\,t) + C_2 \sin(\sqrt{A}\,t)$$

$x = 0$ at $t = 0 \Rightarrow C_1 = 0$

$\dot{x} = v_0$ at $t = 0 \Rightarrow C_2 \sqrt{A} = v_0$

$$\boxed{x = \frac{v_0}{\sqrt{A}} \sin(\sqrt{A}\,t)}$$

(b) $x = C_3 \cosh(\sqrt{-A}\,t) + C_4 \sinh(\sqrt{-A}\,t)$

$x = 0$ at $t = 0 \Rightarrow C_3 = 0$

$\dot{x} = v_0$ at $t = 0 \Rightarrow C_4 \sqrt{-A} = v_0$

$$\boxed{x = \frac{v_0}{\sqrt{-A}} \sinh(\sqrt{-A}\,t)}$$

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4. (a) $W = \int_0^1 dx\, F_x(x, y=0) + \int_0^1 dy\, F_y(x=1, y)$

$= \int_0^1 dx\, x^2 + \int_0^1 dy\,(2y)$

$= \left[\frac{x^3}{3}\right]_0^1 + \frac{2y^2}{2}\Big]_0^1$

$= \frac{1}{3} + 1 = \boxed{\frac{4}{3}}$

(b) $y = x^2$

$\vec{F} \cdot d\vec{r} = F_x\, dx + F_y\, dy$

$= x^2\, dx + (2xy)(2x\, dx)$

$= (x^2 + 4x^4)\, dx$

$\int F \cdot dr = \int_0^1 (x^2 + 4x^4)\, dx$

$= \left[\frac{x^3}{3} + \frac{4x^5}{5}\right]_0^1 = \frac{1}{3} + \frac{4}{5}$

$= \boxed{\frac{17}{15}}$

4. (c) $x = t^3$

$y = t^2$

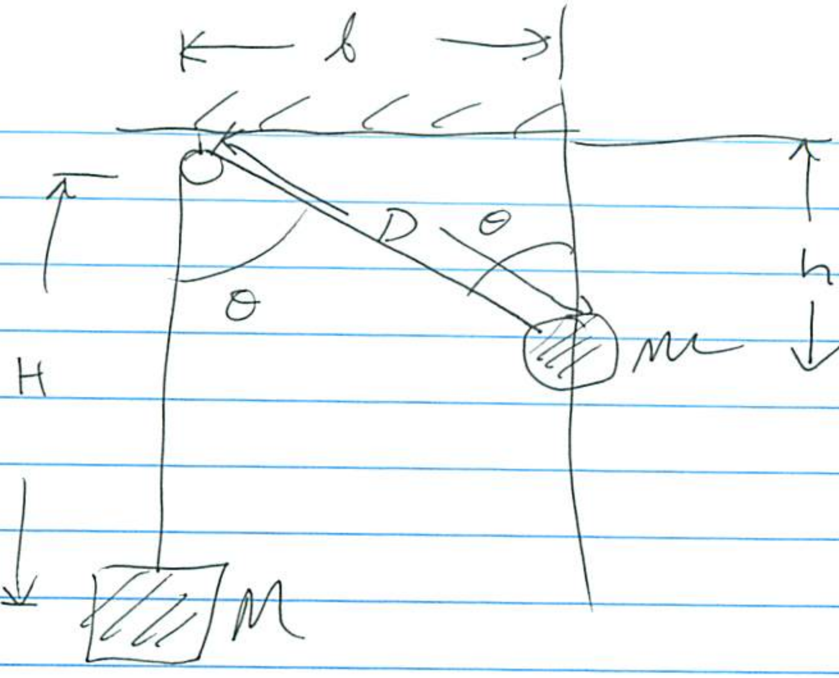
$$F \cdot dr = F_x \, dx + F_y \, dy$$
$$= x^2 (3t^2 dt) + 2xy(2t dt)$$
$$= 3t^8 dt + 4t^6 dt$$
$$\int F \cdot dr = \int_0^1 (3t^8 + 4t^6) dt$$
$$= \left( \frac{3t^9}{9} + \frac{4t^7}{7} \right)_0^1$$
$$= \frac{3}{9} + \frac{4}{7} = \frac{1}{3} + \frac{4}{7}$$
$$= \boxed{\frac{19}{21}}$$

note, $(\text{Curl } F)_z = \frac{\partial}{\partial x} F_y - \frac{\partial F_x}{\partial y}$

not required $= 2y - 0 = 2y \neq 0$

so different paths give different results

5.




$$PE = U = -MgH - mgh$$

$$\frac{b}{h} = \tan\theta$$

$$\frac{b}{D} = \sin\theta$$

$$D + H = l$$

~~$b = h\tan\theta$~~

$$\therefore h = \frac{b}{\tan\theta}$$

$$H = l - D = l - \frac{b}{\sin\theta}$$

$$U = -g(MH + mh) = -g\left[M\left(l - \frac{b}{\sin\theta}\right) + m\frac{b}{\tan\theta}\right]$$

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$$U = -g\left[Ml - \frac{Mb}{\sin\theta} + \frac{mb\cos\theta}{\sin\theta}\right]$$

$$U' = \frac{dU}{d\theta} = -g\left\{\frac{Mb}{\sin^2\theta}\cos\theta + mb\left[\frac{(-\sin\theta)}{\sin\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right]\right\}$$

$$= -gb\left\{M\frac{\cos\theta}{\sin^2\theta} + m\left(-1 - \frac{\cos^2\theta}{\sin^2\theta}\right)\right\}$$

$$= -gb\,\frac{1}{\sin^2\theta}\left(M\cos\theta - m\right)$$

Equilibrium if $\frac{dU}{d\theta} = 0$, i.e. $\cos\theta = \frac{m}{M}$

Since $0 < \theta < 90^\circ$ by the drawing, this solution always exists

if $m < M$

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$$U' = (-gb) \frac{1}{\sin^2\theta} \left( M \cos\theta - m \right)$$

$$U'' = (-gb) \left\{ \frac{-2}{\sin^3\theta} \cos\theta \left( M \cos\theta - m \right) + \frac{1}{\sin^2\theta} \left( -M \sin\theta \right) \right\}$$

$$= (-gb) \frac{1}{\sin^3\theta} \left\{ -2 \cos\theta \left( M \cos\theta - m \right) - M \sin^2\theta \right\}$$

$$= (-gb) \frac{1}{\sin^3\theta} \left\{ -2 \cos\theta \left( M \cos\theta - m \right) - M \left( 1 - \cos^2\theta \right) \right\}$$

at Equ, $\cos\theta = \frac{m}{M}$ so

$$U'' = (-gb) \frac{1}{\sin^3\theta} \left\{ -2 \frac{m}{M} \left( M \frac{m}{M} - m \right) - M \left( 1 - \left(\frac{m}{M}\right)^2 \right) \right\}$$

$$U'' = \frac{-gb}{\sin^3\theta}\left\{(-M)\left(1-\left(\frac{m}{M}\right)^2\right)\right\}$$

$$= \frac{gbM}{\sin\theta} = \frac{gbM}{\sqrt{1-\left(\frac{m}{M}\right)^2}}$$

this is positive, so

Stable Equilibrium if $m<M$

no Equilibrium if $m>M$