Phy 321 Spring 2017
HW 8.1

1. Find minimum of U by

$$\frac{dU}{dr} = 0 \Rightarrow (A)(2)\left(e^{(R-r)/s} - 1\right) e^{(R-r)/s} \left(\frac{-1}{s}\right) = 0$$

$$\Rightarrow e^{(R-r)/s} = 1 \Rightarrow R = r \Rightarrow \boxed{r = R}$$

Near the minimum

$$U(r) = \underbrace{U(R)}_{-A} + \underbrace{U'(R)(r-R)}_{0} + \frac{U''(R)(r-R)^2}{2!}$$

$$U''(r) = \frac{d}{dr}\left\{\left(\frac{-2A}{s}\right)\left[e^{2(R-r)/s} - e^{(R-r)/s}\right]\right\}$$

$$= \left(\frac{-2A}{s}\right)\left[e^{2(R-r)/s}\left(\frac{-2}{s}\right) - e^{(R-r)/s}\left(\frac{-1}{s}\right)\right\}$$

at $r = R$,

$$U''(R) = \left(\frac{-2A}{s}\right)\left(\frac{-2}{s} - \left(\frac{-1}{s}\right)\right) = \frac{2A}{s^2}$$

$$\boxed{U(r) = -A + \left(\frac{A}{s^2}\right)(r-R)^2} + O(r-R)^3$$

$$\frac{1}{2}k(r-R)^2 = \frac{A}{s^2}(r-R)^2 \Rightarrow \boxed{k = \frac{2A}{s^2}}$$

HW 8.2

Easier way: U is obviously minimum at

$$\left(e^{(R-r)/s}-1\right)=0, \text{ i.e. } \boxed{r=R}$$

Let $r-R=x$

$$U = A\left[\left(e^{\frac{x}{s}}-1\right)^2-1\right]$$

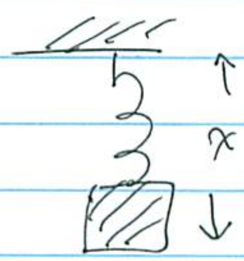
$$= A\left[\left(\left(1+\frac{x}{s}+\frac{1}{2}\left(\frac{x}{s}\right)^2+\cdots\right)-1\right)^2-1\right]$$

$$= A\left[\left(\frac{x}{s}\right)^2-1\right]$$

$$= -A+\frac{A}{s^2}x^2$$

compare with $\text{const}+\frac{1}{2}kx^2 \Rightarrow \boxed{k=\frac{2A}{s^2}}$

2.



$$F = Mg - k(x-l) = M\ddot{x}$$

$-(\text{const})\,\dot{x}$

Let $\omega_0=\sqrt{\frac{k}{M}}$

$$\ddot{x} + \frac{k}{M}(x-\ell) - g + (\text{const})\,\dot{x} = 0$$

initial position is $x = \ell$

final position has $\ddot{x} = \dot{x} = 0$, so

$$\frac{k}{M}\left(x_{\text{Final}} - \ell\right) - g = 0$$

$$\ddot{x} + \frac{k}{M}(x - x_{\text{Final}}) + \text{const}\,\dot{x} = 0$$

let $x - x_{\text{Final}} = y$

$$x = y + x_{\text{Final}}$$

$$\ddot{y} + (\text{const})\,\dot{y} + \frac{k}{M}\,y = 0$$

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2\,y = 0$$

let $y = e^{rt}$ gives $r^2 + 2\beta r + \omega_0^2 = 0$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

critically damped means $\sqrt{\beta^2 - \omega_0^2} = 0$

i.e. $\beta = \omega_0$

$$\ddot{y} + 2\omega_0 \dot{y} + \omega_0^2 y = 0$$

general solution for $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$

is $\left[C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\right] e^{-\beta t}$

where $\omega_1 = \sqrt{\beta^2 - \omega_0^2}$

So for $\beta \to \omega_0$ have $\omega_1 \to 0$ and the two solutions become

$$(C_1 + C_2 t) e^{-\omega_0 t}$$

$$y = e^{-\omega_0 t} (C_1 + C_2 t)$$

$$x = y + x_{Final} = x_{Final} + e^{-\omega_0 t}(C_1 + C_2 t)$$

$$x(0) = x_{Final} + C_1$$

$$0 = \dot{x}(0) \Rightarrow C_2 - \omega_0 C_1 = 0$$

Problem says $x(0) - x_{Final} = 0.5\,m$

So $C_1 = 0.5\,m$, $C_2 = (\omega_0)(0.5\,m)$

HW 8.5

$$x = x_{Final} + (0.5m)(1+\omega_0 t)e^{-\omega_0 t}$$

Also $(k)(0.5m) = Mg \Rightarrow \sqrt{\frac{k}{M}} = \sqrt{\frac{g}{.5m}}$

$$\omega_0 = \sqrt{\frac{g}{0.5m}}$$

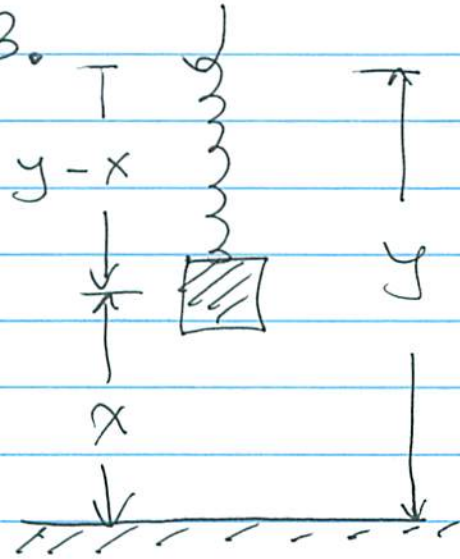
At $t = 1\,sec$, $(x - x_{Final}) = (0.5m)(1+B)e^{-B}$

where $B = \omega_0 t = \sqrt{\frac{g}{0.5m}}(1\,sec)$

use $g = 9.8\frac{m}{sec^2}$ gives $B = 4.4272$

$$\boxed{x - x_{Final} = 0.0324m}$$

3.



$$M\ddot{x} = F = -Mg + k(y - x - \ell)$$

$$\ddot{x} = -g + \omega_0^2 (y - x - \ell)$$

For $t \leq 0$, have

$$\ddot{x} = 0 \quad \text{and}$$

$$x = y_0 - \frac{Mg}{k} = y_0 - \frac{g}{\omega_0^2}$$

and $y = y_0$

$$0 = -g + \omega_0^2 \left(\frac{g}{\omega_0^2} - \ell \right) \Rightarrow \ell = 0$$

$$\ddot{x} + \omega_0^2 x = -g + \omega_0^2 y$$

For $t > 0$

$$\ddot{x} + \omega_0^2 x = -g + \omega_0^2 (y_0 + A \sin \omega t)$$

$$= \omega_0^2 x_0 + \omega_0^2 A \sin \omega t$$

Most general solution:

$$x = x_0 + C_0 \sin(\omega t) + C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

where $(-\omega^2 + \omega_0^2) C_0 = \omega_0^2 A \Rightarrow C_0 = \dfrac{\omega_0^2 A}{-\omega^2 + \omega_0^2}$

$$x = x_0 + \frac{\omega_0^2 A}{-\omega^2+\omega_0^2} \sin(\omega t)$$

$$+ C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

Need $x = x_0$ at $t=0 \Rightarrow C_1 = 0$

Need $\dot{x} = 0$ at $t = 0 \Longrightarrow$

$$\frac{\omega_0^2 A}{-\omega^2+\omega_0^2} \omega + C_2 \omega_0 = 0$$

$$\Rightarrow C_2 = \frac{-\omega_0 A \omega}{-\omega^2+\omega_0^2}$$

(a) $\omega = 2\omega_0$

$$x = x_0 + \left(\frac{-1}{3} A\right) \sin \omega t + \left(\frac{2}{3} A\right) \sin \omega_0 t$$

$$\boxed{x = x_0 - \frac{A}{3} \sin(2\omega_0 t) + \frac{2A}{3} \sin(\omega_0 t)}$$

(b) $\omega = \omega_0$: 1st write

$$x = x_0 + \frac{\omega_0^2 A}{-\omega^2+\omega_0^2} \left[\sin(\omega t) - \frac{\omega}{\omega_0} \sin(\omega_0 t)\right]$$

Use L'Hospital's rule

$$x = x_0 + \frac{\omega_0^2 A}{-2\omega}\left[t\cos(\omega t) - \frac{1}{\omega_0}\sin\omega_0 t\right]$$

now $\omega \to \omega_0$ is ok

$$x = x_0 - \frac{\omega_0 A}{2}\left[t\cos(\omega_0 t) - \frac{1}{\omega_0}\sin\omega_0 t\right]$$

$$\boxed{x = x_0 + \frac{A}{2}\left[\sin(\omega_0 t) - \omega_0 t\cos\omega_0 t\right]}$$

(notice that at resonance with no damping oscillations go to ∞.)

4. $\ddot{x} + x = t(A-t)$

guess particular solution $x = -t^2 + C_A t + C_B$

$\dot{x} = -2t + C_A$

$\ddot{x} = -2$

$\ddot{x} + x = -2 - t^2 + C_A t + C_B = At - t^2$

need $C_A = A$ and $C_B = 2$

$x = -t^2 + At + 2 + C_1 \cos(t) + C_2 \sin(t)$

need $x = 0$ at $t = 0 \Rightarrow C_1 = -2$

need $\dot{x} = 0$ at $t = 0 \Rightarrow A + C_2 = 0$

$$\boxed{x = -t^2 + At + 2 - 2\cos t - A\sin t}$$

could also write as

$$x = t(A-t) + 2(1-\cos t) - A\sin t$$

HW 8.10

5. $\ddot{x} + 2\beta\dot{x} + x = t e^{-\alpha t}$

guess a particular solution :

Try $x = (C_A + C_B t)\, e^{-\alpha t}$

$$\dot{x} = \left[\alpha (C_A + C_B t) + C_B \right] e^{-\alpha t}$$

$$\ddot{x} = \left[-\alpha C_B + (-\alpha)\left[-\alpha (C_A + C_B t) + C_B \right] \right] e^{-\alpha t}$$

need

$$-\alpha C_B - \alpha \left[-\alpha (C_A + C_B t) + C_B \right]$$

$$+ 2\beta \left[-\alpha (C_A + C_B t) + C_B \right]$$

$$+ C_A + C_B t = t$$

coeficient of t : $\alpha^2 C_B - 2\beta\alpha C_B + C_B = 1$

$$\Rightarrow C_B = \frac{1}{\alpha^2 - 2\beta\alpha + 1}$$

coeficient of $\underline{1}$:

$$\Big(-\alpha C_B + \alpha^2 C_A - \alpha C_B$$

$$-2\beta\alpha C_A + 2\beta C_B + C_A \Big) = 0$$

$$2(\beta - \alpha) C_B + (\alpha^2 - 2\beta\alpha + 1) C_A = 0$$

$$C_A = \frac{-2(\beta-\alpha)C_B}{\alpha^2 - 2\beta\alpha + 1}$$

$$= \frac{-2(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2}$$

then add the general solution to homogeneous Eq.:

$$x = \left[\frac{-2(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2} + \frac{t}{(\alpha^2-2\beta\alpha+1)}\right]e^{-\alpha t}$$

$$+\left[C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\right]e^{-\beta t}$$

where ~~β~~ $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

Now we need $x = 0$ and $\dot{x} = 0$ at $t = 0$ to determine C_1 and C_2

Easy Way: Expand x at $t=0$, keeping only const and t^1 :

HW 8.12

$$x = \frac{-2(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2}(1-\alpha t)$$

$$+ \frac{t}{\alpha^2-2\beta\alpha+1}$$

$$+ c_1(1-\beta t) + c_2\omega_1$$

coeffient of t $\quad \dfrac{(-2)(\beta-\alpha)(-\alpha)}{(\alpha^2-2\beta\alpha+1)^2}$

$$+ \frac{1}{\alpha^2-2\beta\alpha+1} - \beta c_1 = 0$$

$$\Rightarrow c_1 = \frac{1}{\beta}\left\{\frac{2\alpha(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2} + \frac{1}{\alpha^2-2\beta\alpha+1}\right\}$$

coefficient of 1

$$\frac{-2(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2} + c_1 + c_2\omega_1 = 0$$

where $\omega_1 = \sqrt{\omega_0^2-\beta^2}$
$= \sqrt{1-\beta^2}$

Can simplify

$$C_1 = \left(\frac{1}{\beta}\right)\frac{1}{(\alpha^2-2\beta\alpha+1)^2}\Big\{$$

$$2\alpha(\beta-\alpha)+\alpha^2-2\beta\alpha+1\Big\}$$

where $\{\,\} = -\alpha^2+1$

$$C_1 = \left(\frac{1}{\beta}\right)\frac{1-\alpha^2}{(\alpha^2-2\beta\alpha+1)^2}$$

$$C_2 = \left(\frac{-1}{\omega_1}\right)\left[\frac{-2(\beta-\alpha)}{(\alpha^2-2\beta\alpha+1)^2} + C_1\right]$$

$$= \frac{-1}{\sqrt{1-\beta^2}}\,\frac{1}{(\alpha^2-2\beta\alpha+1)^2}\Big\{$$

$$-2(\beta-\alpha)+\frac{1}{\beta}(1-\alpha^2)\Big\}$$

$$= \frac{-1}{\sqrt{1-\beta^2}}\,\frac{1}{(\alpha^2-2\beta\alpha+1)^2}\,\frac{1}{\beta}\Big\{$$

$$-2\beta^2+2\beta\alpha+1-\alpha^2\Big\}$$