Section 3.2

Rockets

Read Section 3.2.

The rocket contains a fuel that burns rapidly. As the exhaust gas is expelled from the combustion chamber, there is a reaction force on the rocket; ***"thrust".***

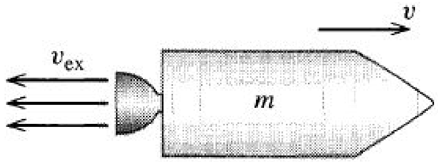


Figure 3.2 A rocket of mass m travels to the right with speed v and ejects spent fuel with exhaust speed v_{ex} relative to the rocket.

Now analyze the motion, using the principle of momentum, i.e.,

$$\dot{\mathbf{P}} = \mathbf{F}^{ext} \quad . \qquad \textit{eq. (3.1)}$$

But be careful to identify **P** correctly!

(Dot ˙ means d/dt)

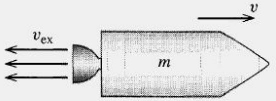


Figure 3.2 A rocket of mass m travels to the right with speed v and ejects spent fuel with exhaust speed v_{ex} relative to the rocket.

<u>Derive the equation of motion for the rocket.</u>

What I mean by the "rocket" is the metal cylinder plus the fuel *<u>inside</u>*.

The mass of the rocket decreases as fuel is expelled out the back.

Let m(t) = the mass of the rocket at time t;

$$m(t) = M_{cyl} + M_{fuel\ inside}(t) \quad ;$$

The equation of motion will depend on two parameters of the rocket engine.

■ Let v_{ex} = the *relative speed* of the exhaust gas. Understand this:

The velocity of the exhaust gas at time t is $v(t) - v_{ex}$, where v(t) is the velocity of the rocket.

■ Let K = the *mass rate* of the exhaust gas. *K is positive.* Understand this:

$$K = - dM_F / dt = - dm/dt$$

I.e., K δt = the mass of fuel expelled by the engine during the time δt

= the *decrease* of the rocket mass during δt. (units of K: kg/s)

Consider the change of momentum of the system = *rocket and fuel* , from time t to time $t + \delta t$.

The total momentum at time t is

$$P(t) = m(t)\, v(t) + P_{\text{fuel already expelled before t}}$$

(I'm only considering one-dimensional motion of the rocket.)

The total momentum at time $t + \delta t$ is

$$P(t+\delta t) = m(t+\delta t)\, v(t+\delta t)$$ ←rocket at t+dt

$$+ (K\,\delta t)\,[\, v(t) - v_{ex} \,]$$ ←fuel expelled during dt

$$+ P_{\text{fuel already expelled before t}}$$

$$= (m + \delta m)(v + \delta v) + (K\,\delta t)\,(v - v_{ex}) + P_{\text{already}}$$

$\delta m = -K\,\delta t$

DERIVE IT YOURSELF

(Because I will take δt to be infinitesimal, it does not matter if I take the exhaust speed during δt to be $v(t)-v_{ex}$ or $v(t+\delta t)-v_{ex}$ or something in between.)

The *change* of total momentum = δP

$$= P(t+\delta t) - P(t) .$$

Now, the equation of motion follows from the theorem, $dP/dt = F^{ext}$.

$$\therefore F^{ext} = \lim\, [P(t+\delta t) - P(t)]/\delta t \qquad (\delta t \to 0)$$

$$= [\, (m - K\,\delta t)\,(v+\delta v) + K\,\delta t\,(v - v_{ex}) - mv \,]/\delta t$$

$$= [\, m\,\delta v - K\,\delta t\, v + K\,\delta t\,(v - v_{ex}) + O(\delta^2)]/\delta t$$

$$= m\, dv/dt - K\, v_{ex} \qquad \text{in the limit } \delta t \to 0.$$

Note the cancellations!

The rocket equations

For a rocket moving in one dimension, the velocity v(t) obeys

$$m \frac{dv}{dt} = K v_{ex} + F^{ext} \qquad (1)$$

where v_{ex} = the relative speed of the exhaust and K = the mass rate of the exhaust. Here m(t) is the mass of the rocket including enclosed fuel, so

$$\frac{dm}{dt} = -K \qquad (2)$$

Thrust force = $K v_{ex}$.

Example. A rocket in deep space with constant values of v_{ex} and K ...

Let the initial velocity be v_0 and the initial mass be m_0.

Calculate the velocity at time t.

By eq.(2), $m(t) = m_0 - K t$

Then eq.(1) is

$$(m_0 - K t)\,(dv/dt) = K v_{ex}$$

Using separation of variables,

$$dv = K v_{ex} / (m_0 - K t)\; dt \,;$$

integrate both sides of the equation,

$$v - v_0 = - v_{ex} [\, \ln (m_0 - K t) - \ln(m_0) \,]$$

$$v = v_0 + v_{ex} \ln [\, m_0 / (m_0 - K t)]$$

Example #2. A rocket in deep space with constant v_{ex} and arbitrary time dependence , K(t)

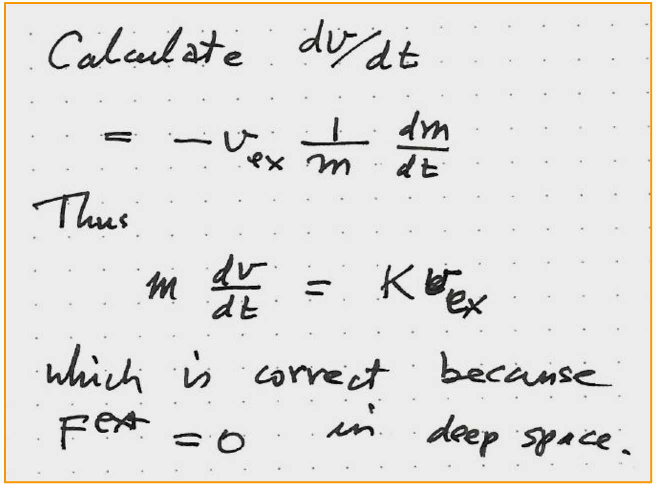
Let the initial velocity be v_0 and the initial mass be m_0.

Calculate the velocity at time t.

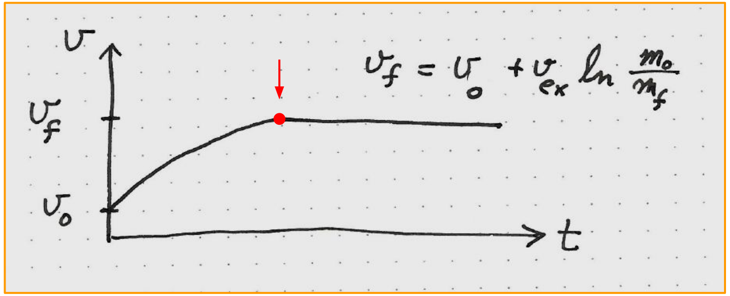
Result: $v = v_0 + v_{ex}\, ln\, [m_0\ /m(t)\ ]$

Proof:

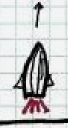
$$v = v_0 + v_{ex}\, ln\, [m_0] - v_{ex}\, ln\, [\ m(t)\ ]$$



Graph of velocity versus time for constant K .



Take off from Earth's surface



$$m \frac{dv}{dt} = K v_{ex} - mg$$

Approximate g = constant and v_{ex} = constant.

Separation of variables

$$m\, dv = K v_{ex}\, dt - mg\, dt$$

$$dv = v_{ex}\left(\frac{-dm}{m}\right) - g\, dt$$

$$dv' = v_{ex}\left(\frac{-dm'}{m'}\right) - g\, dt'$$

Integrate

t' = intermediate time $\in (0, t)$

$$\int_0^v dv' = v = \int_{m_0}^{m} v_{ex} \frac{(-dm')}{m'} - \int_0^t g\, dt'$$

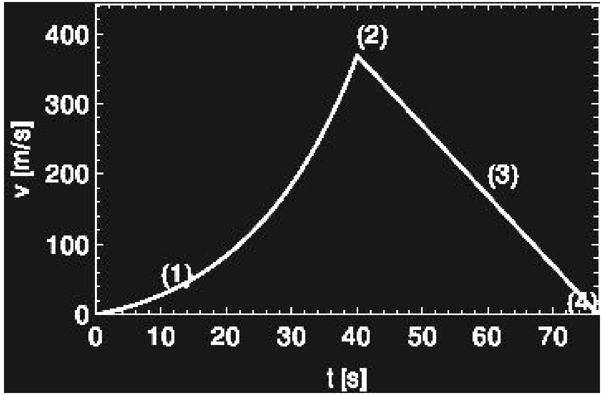
$$v(t) = v_{ex} \ln \frac{m_0}{m(t)} - gt$$

Also, $m(t) = m_0 - Kt$ if K is constant.

A successful take-off requires

$K v_{ex} > mg$ i.e. thrust > weight

Graph of v as a function of t



(1) slope = acceleration

(2) runs out of fuel

(3) still rising but a = -g

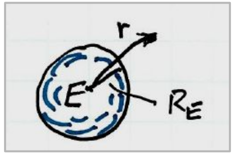
(4) max height = area under the curve

A more difficult example:

Fire a rocket to the stars

Now g is not constant. As the distance from the earth increases, g decreases;

$$g(r) = g_s \, R_E^{\,2} / r^2 \qquad \text{for} \quad r > R_E \, .$$



The equation of motion is

$$m \frac{dv}{dt} = K v_{ex} - m g_s \frac{R_E^2}{r^2}$$

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TAKE OFF FROM EARTH'S SURFACE

We have

$m\,(dv/dt) = K\,v_{ex} - mg$

Approximate

g = constant and v_{ex} = constant.

How to integrate the diff. eq.?

$$m\,dv = K\,v_{ex}\,dt - mg\,dt$$

but m = m(t) !

$$= -v_{ex}\,dm - m\,g\,dt$$

$$dv = -v_{ex}\,(dm/m) - g\,dt$$

$dv' = -v_{ex}\,(dm'/m') - g\,dt'$

Integrate

$$\int_0^v dv' = -v_{ex}\int_{m0}^{m} dm'/m' - g\int_0^t dt'$$

$$v(t) = v_{ex}\,\ln\,[m_0/m(t)] - gt$$

Also, $m(t) = m_0 - Kt$ ***(assuming K is constant)***

Graph of v as a function of t

(1) Slope ~ $(v_{ex}K - m_0g)/m_0$

(2) rocket runs out of fuel

(3) slope = $-g$

(4) rocket starts to fall downward

Area under the curve = max. height