PHYSICS 422/820 FINAL EXAM Dec 16, 2016 NAME ____________________

1. [10 pts] Find the normal mode oscillation frequencies for a system whose coordinates x and y obey

$$\begin{aligned} \ddot{x} + \ddot{y} &= -2\,x \\ \ddot{x} + 2\,\ddot{y} &= -3\,y \end{aligned}$$

2. [10 pts] The Lagrangian for a certain mechanical system is given by

$$L = \frac{1}{2}\left[\dot{r}^2 + r^2(\dot{\theta} + \omega)^2\right] - r^2 \cos(2\,\theta)$$

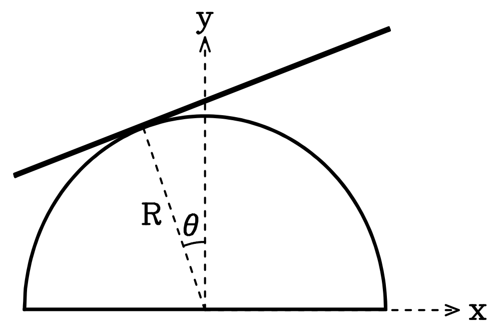
where r and θ are the generalized coordinates and ω is a constant.

(a) Find the Hamiltonian as a function of its correct ("canonical") variables.

(b) Use the Hamiltonian to obtain the equations of motion.

3. [10 pts] A uniform thin stick of length ℓ and mass M rocks back and forth on top of a half-cylinder which does not move. When $\theta = 0$, the stick is centered on the top of the cylinder. It does not slip, so the point on the stick that is touching the cylinder in the drawing is a distance $R\theta$ from the center of the stick. The stick moves only in the plane perpendicular to the cylinder. Find the Lagrangian as a function of θ, $\dot{\theta}$, M, ℓ, R.

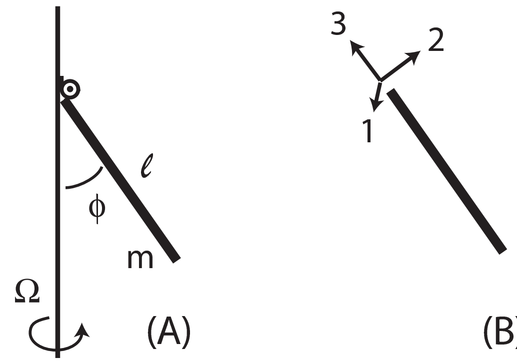
   Hints: You may find it helpful to find the position of the center of mass in terms of the coordinates x and y. The moment of inertia of the stick about its center is $M\ell^2/12$. You only need to find the Lagrangian; you are not required to find or solve the equations of motion.



4. [10 pts] A centrifugal speedometer consists of a uniform cylindrical rod of mass m, length ℓ, and radius r, which is attached by means of a hinge to a vertical axle that is rotating at a fixed angular velocity Ω as shown in Fig. (A).

The hinge allows the angle ϕ to change freely, but forces the azimuthal angle of the rod to rotate with angular velocity Ω about the vertical direction.

Let $I_1 = I_2$ and I_3 be the moments of inertia of the rod for rotations about its endpoint, about its principal axis directions. Direction 3 points along the symmetry axis of the rod, direction 2 points perpendicular to the rod in the vertical plane, and direction 1 is horizontal ("out of the paper"), as shown in Fig. (B).



(A) (B)

(a) Find the Lagrangian for this system as a function of ϕ, $\dot{\phi}$, and constants Ω, $I_1 = I_2$, I_3, m, g, ℓ.

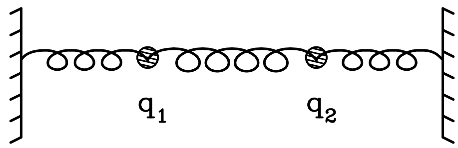
(b) Find the equilibrium angle ϕ (i.e., the angle at which ϕ will stay constant) as a function of Ω and the other constants in the problem.

5. [10 pts] Two particles of equal mass M are attached to the junctions of three identical springs as shown in the figure. The masses move only in the horizontal direction. The three springs each have unstretched length zero and spring constant K. Let q_1 and q_2 be the coordinates of the two masses, measured to the right from their equilibrium positions.

(a) Find the eigenfrequencies and normal modes of the system.

(b) Determine the particle positions as a function of time if at time $t = 0$, both masses are at their equilibrium positions and particle 1 has velocity v and particle 2 has velocity 0.

Hint: you can use symmetry to figure out what the normal modes must be.



6. [10 pts] The moment of inertia tensor of a rigid body, in the frame in which it is diagonal, is given by

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Find the moment of inertia tensor of this body in the coordinate frame that is obtained by rotating it about the z axis through an angle of $30°$.

7. [10 pts] Under a rotation, the components of a vector such as the angular momentum of a rigid body change by

$$L'_i = \sum_j A_{ij}\, L_j$$

where L'_1, L'_2, L'_3 are the new components of the vector, and L_1, L_2, L_3 are its old components, and the rotation matrix A is orthogonal.

(a) Derive the corresponding expression that relates the new components of the moment of inertia tensor I'_{ij} to the old ones. Your answer will involve the elements A_{ij} of the arbitrary rotation matrix.

(b) Use your result from part (a) to prove that the trace of the moment of inertia tensor is not changed by any rotation.

8. [10 pts] *Not all courses in classical mechanics cover continuum mechanics, so this last problem is not part of the Subject Exam. If you are not currently enrolled in PHY422 or PHY820, you do not need to do it. But if you are taking PHY422 or PHY820 for credit, this problem will contribute to your course grade—along with the rest of this exam, your previous exams, and your homework scores.*

With a convenient choice of units, the Lagrangian for a particular continuous system is

$$L = \frac{1}{2} \int_0^1 \left[y^2 + x \left(\frac{\partial y}{\partial t} \right)^2 - x^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx$$

(a) Find the equation of motion, which is a partial differential equation for $y(x, t)$.

(b) Assume an oscillating solution to the equation of motion and find the resulting ordinary differential equation for the x-dependence of the motion.