Physics 422/820 – Fall 2016

Homework #13, Due at beginning of class Friday Dec 9.

1. [6 pts] A particle of mass M moves under the influence of a central potential $V(r)$ where $r = |\vec{r}|$. The particle is found to move along the curve $r^2 = a^2 \cos(2\phi)$, where r and ϕ are the usual polar coordinates and a is a constant.

   (a) Find $V(r)$. (As you know, you can always add an arbitrary constant to any potential energy function. Use that freedom to make $V(r) \to 0$ in the limit $r \to \infty$. After doing that, your answer for $V(r)$ will contain a single free parameter that corresponds to the angular momentum.)

   (b) Find the time it takes for the particle to travel from $r = a$ to $r = 0$. (Your answer will involve the same constants as your answer to part (a).

2. [6 pts] A thin flexible string of length a has a linearly varying mass density: its mass per unit length is $\rho(x) = bx^2$ where $0 < x < a$ and b is a positive constant. It is stretched between two support points, such that it is under a constant tension τ, and its displacement $y(x,t)$ is always zero at $x = 0$ and $x = a$. It vibrates in a plane, and you can assume the usual small-oscillations approximations. Gravity can be neglected.

   (a) Find the differential equation that must be solved to obtain its normal mode frequencies.

   (b) Find the frequencies of each of its three lowest-frequency modes. You will need to use some numerical methods to solve this. (Special Mathematica hint: that program gives you different forms of the solution depending on whether you write the power law as x^2 or $x^{2.}$; and one of those forms is more convenient than the other.)

3. [6 pts] A uniform has mass m, spring constant k, and unstretched length a. One end of the spring is attached to a wall, so it is always at $x = 0$. The other end of the spring is attached to a point mass M which is free to slide without friction on a horizontal track.

   (a) Let $x(s,t)$ define the position of the spring as a function of time, with $0 < s < a$. Find the Lagrangian density for the spring, and use it to find the equation of motion for the spring.

   (b) Solve the equation of motion for the spring corresponding to oscillation at frequency ω, subject to the boundary condition $x(0,t) = 0$ at all t.

   (c) Impose the boundary condition corresponding to $F = Ma$ for the point mass M, to derive an equation that determines the frequencies of the natural modes of vibration of this system.

   (d) Find the frequencies of each of the two lowest frequency modes of vibration of this system for the case $m = M$. (You will need to do this last part numerically.)

(Last updated 12/3/2016.)

Phy 422/820 Fall 2016

HW 13.1

1. $\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{const since } \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$H = \text{const since } \frac{\partial \mathcal{L}}{\partial t} = 0$$

Here, $H = T + V$ so

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\left[\frac{p_\phi^2}{2mr^2} + V(r)\right]}_{V_{eff}(r)}$$

$$\dot{r} = \pm\sqrt{\frac{2}{m}}\sqrt{\left[E - V_{eff}(r)\right]}$$

$$\dot{\phi} = \frac{p_\phi}{mr^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{\dot{r}}{\dot{\phi}}\right)^2 = \frac{2r^4 m}{p_\phi^2}\left(E - \frac{p_\phi^2}{2mr^2} - V(r)\right) \quad \text{---}(1)$$

(true for any spherically symm.)

$$r^2 = a^2 \cos 2\phi$$

$$2r\,dr = a^2(-2\sin(2\phi)\,d\phi)$$

$$\left(\frac{dr}{d\phi}\right) = \frac{-a^2}{r}\sin(2\phi)$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{a^4}{r^2}\left[\sin(2\phi)\right]^2 = \frac{a^4}{r^2}\left[1-\cos^2 2\phi\right]$$

$$= \frac{a^4}{r^2}\left[1-\frac{r^4}{a^4}\right] = \frac{a^4}{r^2} - r^2$$

$$\frac{a^4}{r^2} - r^2 = \frac{2\mu r^4}{p_\phi^2}\left(E - \frac{p_\phi^2}{2\mu r^2} - V(r)\right)$$

$$\frac{p_\phi^2}{2\mu r^4}\left(\frac{a^4}{r^2} - \cancel{r^2}\right) = E - \cancel{\frac{p_\phi^2}{2\mu r^2}} - V(r)$$

$$V(r) = E - \cancel{\frac{p_\phi^2}{2\mu r^2}} - \frac{p_\phi^2 a^4}{2\mu r^6} + \cancel{\frac{p_\phi^2}{2\mu r^2}}$$

can add any constant to V, so

(a) $$\boxed{V(r) = -\frac{p_\phi^2 a^4}{2\mu}\frac{1}{r^6}}$$

1. (b) Easy way: $p_\phi = \mu r^2 \dot\phi$

$$= \mu a^2 \cos(2\phi)\,\dot\phi$$

$$\frac{d\phi}{dt} = \frac{p_\phi}{\mu a^2 \cos(2\phi)}$$

$$\int d\phi \cos(2\phi) = \int \frac{p_\phi}{\mu a^2}\,dt$$

$$\frac{1}{2}\sin(2\phi) = \frac{P_\phi}{Ma^2}t + \text{const}$$

Let $r = a \Rightarrow \phi = 0$ at $t = 0$

$\Rightarrow$ const $= 0$

$$\sin(2\phi) = \frac{2P_\phi}{Ma^2}t$$

Reaches $r = 0$ at $\cos(2\phi) = 0 \Rightarrow \phi = \frac{\pi}{4}$

at time $\sin\left(\frac{\pi}{2}\right) = \frac{2P_\phi}{Ma^2}t$

$$\Rightarrow \boxed{t = \frac{Ma^2}{2P_\phi}}$$

Hard Way:

$$E = \frac{1}{2}M\dot{r}^2 + \frac{P_\phi^2}{2Mr^2} + V(r)$$

at $t = 0$, $\dot{r} = -\frac{a^2}{r}\underbrace{\sin(2\phi)}_{\to 0}\dot{\phi}$

so

$$E = \frac{1}{2}M\dot{r}^2 + \frac{P_\phi^2}{2Mr^2} - \frac{P_\phi^2 a^4}{2M}\frac{1}{r^6}$$

with $E = 0$

$$\dot{r} = \pm\sqrt{\frac{2}{M}}\sqrt{\underbrace{E}_{=0} - V_{eff}(r)}$$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{M}}\sqrt{-\frac{p_\phi^2}{2Mr^2} + \frac{p_\phi^2 a^4}{2Mr^6}}$$

$$= \pm\sqrt{\frac{2}{M}}\sqrt{\frac{p_\phi^2}{2M}}\sqrt{\frac{a^4}{r^6} - \frac{1}{r^2}}$$

$$\int \frac{dr}{\sqrt{\frac{a^4}{r^6} - \frac{1}{r^2}}} = \pm\int \frac{|p_\phi|}{M}\,dt + \text{const}$$

$$\int \frac{r\,dr}{\sqrt{\frac{a^4}{r^4} - 1}} = \frac{p_\phi}{M}t + \text{const}$$

$$-\frac{a^2}{2}\sqrt{1 - \frac{r^4}{a^4}} = \frac{p_\phi}{M}t \underbrace{+\ \text{const}}_{\to 0}$$

$$\frac{a^4}{4}\left(1 - \frac{r^4}{a^4}\right) = \left(\frac{p_\phi}{M}t\right)^2$$

$$1 = \frac{r^4}{a^4} + \left(\frac{2p_\phi}{Ma^2}t\right)^2$$

$$r = a\left[1 - \left(\frac{2p_\phi}{Ma^2}\right)^2 t^2\right]^{1/4}$$

Hits $r = 0$ at $t = \dfrac{Ma^2}{2p_\phi}$ ✓

2. Part of string from x to $x+dx$

is stretched by $\sqrt{dx^2+dy^2}-dx$

$$= dx\left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}-1\right] = dx\,\frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

Potential Energy is the work needed to do that:

$$V = \int_0^a dx\,\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \tau$$

Kinetic Energy is "$\frac{1}{2}mv^2$" $= \int_0^a \frac{dx}{2}\,\rho(x)\left(\frac{\partial y}{\partial t}\right)^2$

$$\mathcal{L}_{density} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$$

$$\delta\int L\,dt = \delta\int \mathcal{L}_{density}\,dx\,dt = 0$$

$$\Rightarrow \frac{\partial}{\partial y}\mathcal{L}_{density} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}_d}{\partial\left(\frac{\partial y}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}_d}{\partial\left(\frac{\partial y}{\partial x}\right)}$$

$$0 = \frac{\partial}{\partial t}\rho\left(\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\tau\left(\frac{\partial y}{\partial x}\right)$$

$$0 = \rho(x)\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2}$$

HW 13.6

Let $y(x,t) = A(x)\cos\omega t$

$$0 = \left[\rho(x)A(x)(-\omega^2) - \tau \frac{d^2A}{dx^2}\right]\cancel{\cos\omega t}$$

$$\frac{d^2A}{dx^2} + \frac{\rho\omega^2}{\tau}A = 0$$

where $\rho = bx^2$

$$\frac{d^2A}{dx^2} + \left(\frac{b}{\tau}\omega^2\right)x^2 A = 0$$

In $a=b=\tau=1$ units,

$y = \text{const} \times J_{\frac{1}{4}}\left(\frac{\omega}{2}x^2\right)$ by mathematica

Need $y=0$ at $x=1$

$\Rightarrow \omega_1 = 5.56178$

$\omega_2 = 11.8123$

$\omega_3 = 18.0848$

in UNITS where $a=b=\tau=1$

$\frac{1}{x^2} \doteq \frac{bx^2\omega^2}{\tau}$ in UNITS, so

$$\frac{1}{a^2} = \frac{\sigma a^2 \omega^2}{\tau}$$

$$\omega^2 = \frac{\tau}{\sigma a^4} \Rightarrow \omega = \sqrt{\frac{\tau}{\sigma a^4}}$$ in UNITS

so
$$\boxed{\begin{aligned} \omega_1 &= 5.56 \sqrt{\frac{\tau}{\sigma a^4}} \\ \omega_2 &= 11.81 \sqrt{\frac{\tau}{\sigma a^4}} \\ \omega_3 &= 18.08 \sqrt{\frac{\tau}{\sigma a^4}} \end{aligned}}$$

HW13.8

HW13Fall2016problem2

```
In[666]:= (* For Problem 2, in units where a = b = tau = 1 *)

In[667]:= DSolve[{A''[x] + omega^2 * x^2. * A[x] == 0, A[0] == 0, A'[0] == 1}, A[x], x]
```

Out[667]= $\{\{A[x] \to (1.28185\sqrt{x}\ \text{BesselJ}[0.25, 0.5\ \text{omega}\ x^2]) / \text{omega}^{1/4}\}\}$

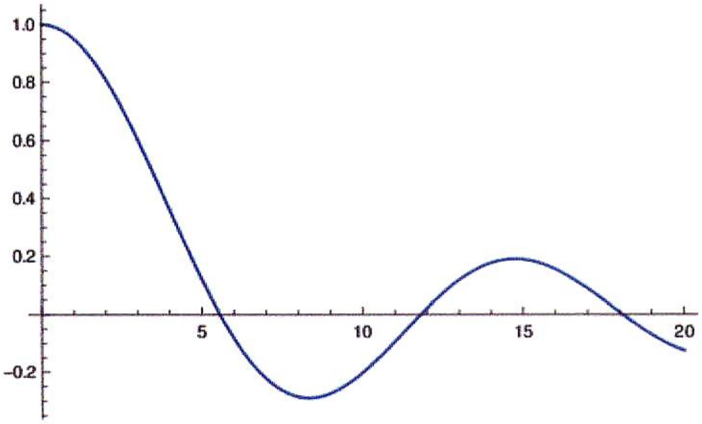
```
In[668]:= y = A[x] /. %[[1]]
```

Out[668]= $(1.28185\sqrt{x}\ \text{BesselJ}[0.25, 0.5\ \text{omega}\ x^2]) / \text{omega}^{1/4}$

```
In[669]:= yend = y /. x → 1
```

Out[669]= $(1.28185\ \text{BesselJ}[0.25, 0.5\ \text{omega}]) / \text{omega}^{1/4}$

```
In[670]:= Plot[yend, {omega, 0, 20}, PlotRange → All]
```

Out[670]=


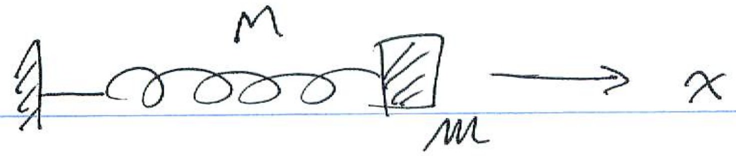

```
In[671]:= FindRoot[yend == 0, {omega, 5}]
Out[671]= {omega → 5.56178}

In[672]:= FindRoot[yend == 0, {omega, 12}]
Out[672]= {omega → 11.8123}

In[673]:= FindRoot[yend == 0, {omega, 18}]
Out[673]= {omega → 18.0848}

In[674]:= (* Actually, Mathematica knows the zeros of the Bessel functions: *)

In[675]:= For[i = 1, i ≤ 3, i++, Print[N[2 * BesselJZero[1 / 4, i]]]]
5.56178
11.8123
18.0848
```

3. 

(a) For the Spring,

$$T = ``\tfrac{1}{2} m v^2" = \frac{1}{2}\left(\frac{M}{a}\,ds\right)\left(\frac{\partial x}{\partial t}\right)^2$$

$$V = ``\tfrac{1}{2} k (\Delta x)^2" = \int \frac{1}{2}\left(\frac{ka}{ds}\right)\left[x(s+ds) - x(s) - ds\right]^2$$

$$= \int \frac{1}{2}\frac{ka}{ds}\left[\cancel{x(s)} + ds\,x'(s) - \cancel{x(s)} - ds\right]^2$$

$$= \int \frac{1}{2} k\,a\,ds\,(x' - 1)^2$$

$$= \int \frac{1}{2} k\,a\,ds\left(\frac{\partial x}{\partial s} - 1\right)^2$$

$$\mathcal{L}_{dens} = \frac{1}{2}\frac{M}{a}\left(\frac{\partial x}{\partial t}\right)^2 - \frac{1}{2} ka\left(\frac{\partial x}{\partial s} - 1\right)^2$$

$$L = \int \mathcal{L}_{dens}\, ds$$

$$0 = \int L\, dt = \int \mathcal{L}_{dens}\, ds\, dt$$

Euler-Lagrange Eq. $\dfrac{\partial \mathcal{L}_{dens}}{\partial x} = \dfrac{\partial}{\partial t}\dfrac{\partial \mathcal{L}_{dens}}{\partial\left(\frac{\partial x}{\partial t}\right)} + \dfrac{\partial}{\partial s}\dfrac{\partial \mathcal{L}_{dens}}{\partial\left(\frac{\partial x}{\partial s}\right)}$

$$\Rightarrow 0 = \frac{\partial}{\partial t}\frac{M}{a}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial s}(-ka)\left(\frac{\partial x}{\partial s} - 1\right)$$

HW 13.10

$$\boxed{0 = \frac{M}{a}\frac{\partial^2 x}{\partial t^2} - ka\frac{\partial^2 x}{\partial s^2}}$$

Static solution $\frac{\partial^2 x}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 x}{\partial s^2} = 0$

$$\Rightarrow x = C_1 + C_2 s$$

$x = 0$ at $s = 0$, so $x = C_2 s$

Note, in static solution,

$$V = \frac{1}{2} ka \int ds \left(\frac{dx}{ds} - 1\right)^2$$

$$= \frac{1}{2} ka (C_2 - 1)^2 a$$

Minimum $\Rightarrow C_2 = 1$ so $x = s$ is Equilibrium

Oscillating solution is (or $e^{i\omega t}$ or $\sin\omega t$)

$$x = s + A(s)\cos(\omega t)$$

$$0 = \frac{M}{a}(-\omega^2)A - ka\frac{d^2A}{ds^2}$$

$$\frac{d^2A}{ds^2} + \frac{M\omega^2}{ka^2}A = 0$$

HW 13, 11

$$A = C_3 \sin\left(\sqrt{\frac{M\omega^2}{ka^2}}\, s\right) + C_4 \cos\left(\sqrt{\frac{M\omega^2}{ka^2}}\, s\right)$$

Need $A = 0$ at $s = 0 \Rightarrow C_4 = 0$

$$\boxed{\begin{aligned} A(s) &= C_3 \sin\left(\sqrt{\frac{M\omega^2}{ka^2}}\, s\right) \\ x &= A(s)\cos\omega t + s \end{aligned}}$$

(c) the right hand end of spring is at $s = a$

$$x(a, t) = a + C_3 \sin\sqrt{\frac{M\omega^2}{k}}\, \cos\omega t$$

Hence "$F = ma$" for mass m gives

$$F = m\ddot{x} = m(-\omega^2)\, C_3 \sin\sqrt{\frac{M\omega^2}{k}}\, \cos(\omega t)$$

Also, the tension in the Spring is

$$F = \text{"}-kx\text{"} = -\frac{ka}{ds}\left[x(s+ds) - x(s) - ds\right]$$

$$= \frac{-ka}{ds}\left(x'(s) - 1\right) ds$$

$$= -ka\left(\frac{\partial x}{\partial s} - 1\right) \text{ at } s$$

$$\cancel{-ka\left[C_3\sqrt{\frac{M\omega^2}{ka^2}}\cos\left(\sqrt{\frac{M\omega^2}{ka^2}}\, s\right)\right]}$$

$$F = -ka\, C_3 \sqrt{\frac{M\omega^2}{ka^2}} \cos\sqrt{\frac{M\omega^2}{k}} \cos\omega t$$

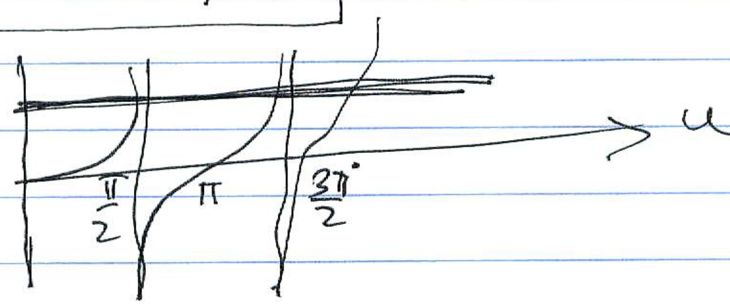
Equate this to F above

$$m(-\omega^2)\, C_3 \sin\sqrt{\frac{M\omega^2}{k}} \cos\omega t$$
$$= -ka\, C_3 \sqrt{\frac{M\omega^2}{ka^2}} \cos\sqrt{\frac{M\omega^2}{k}} \cos\omega t$$

$$m\omega^2\ \cancel{a} \tan\sqrt{\frac{M\omega^2}{k}} = k\sqrt{\frac{M\omega^2}{k}}$$

Let $u = \sqrt{\frac{M\omega^2}{k}}$

Then $\boxed{u \tan u = \frac{M}{m}}$



For $M = m$, need $u \tan u = 1$

lowest: $u = 0.8603$, so $\omega = \sqrt{\frac{k}{M}}\, 0.8603$

Next lowest: $u = 3.426$, so $\omega = \sqrt{\frac{k}{m}}(3.426)$

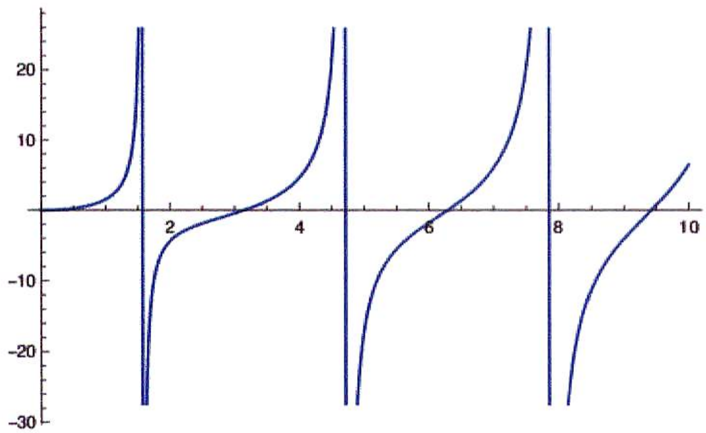
HW 13.13

HW13Fall2016problem3

```
y = u * Tan[u]
```

u Tan[u]

```
Plot[y, {u, 0, 10}]
```




```
FindRoot[y == 1, {u, 1}]
```

{u → 0.860334}

```
FindRoot[y == 1, {u, 3}]
```

{u → 3.42562}