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Ultrahigh-Energy Neutrino-Nucleon Cross Sections and Perturbative Unitarity

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Abstract

Unitarity relates the total cross section for neutrino-nucleon scattering to the neutrino-nucleon forward scattering amplitude. Assuming the validity of the perturbative expansion of the forward amplitude in the *weak* coupling constant, we derive a unitarity bound on the inelastic cross section. The inelastic cross section saturates this bound at a typical neutrino energy $E_\nu \simeq 10^8$ GeV. This implies that calculations of the inelastic cross section that use current parton distribution functions and lowest order weak perturbation theory are unreliable above this energy.

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I. INTRODUCTION

The phenomenology of ultrahigh-energy (UHE) neutrinos and their detection [1] depends on the neutrino-nucleon total cross section, which has been calculated in the standard model in Refs. [2–6]. A striking feature of all of these predictions is the continued power-law-like growth of the cross sections with E_ν at the highest energies. This rise with E_ν is directly related to the very-low-x behaviour of the nucleon parton distribution functions (PDF's), which must be extrapolated below the regime of the current HERA data [7]. It has even been argued [8–11] that cosmic ray events will soon require a cross section even larger than that given by the standard calculations.

On the other hand, unitarity relates the total scattering cross section to the forward scattering amplitude. For neutrino-nucleon scattering, in contrast to electron-nucleon scattering, the forward amplitude can be determined and used to bound the total cross section. This bound, based on lowest order perturbation theory in the weak coupling, is independent of the PDF's and implies that the inelastic cross section cannot rise indefinitely. In fact, it cannot rise for more than about two decades in energy above the region now covered by HERA. Thus, either there will need to be a dramatic change in the way parton distribution functions scale not too far above the HERA data, or the assumption of weak perturbation theory is wrong. In either case the inelastic cross section will likely be very different at high E_ν than what is predicted in the references above.

In Section 2 we discuss the relation between the cross section at high E_ν and the low x behavior of the distribution functions. In Section 3 we derive our bound and Section 4 contains conclusions.

II. ULTRAHIGH-ENERGY NEUTRINO-NUCLEON CROSS SECTIONS

At ultrahigh neutrino energies the total cross section is completely dominated by its deep inelastic component. The deep inelastic scattering (DIS) cross section can be evaluated within the QCD-improved parton model, employing the universal parton distribution functions (PDF's) of the nucleon [12–14] which are derived from fitting photon-exchange dominated DIS measurements. Most important for high energies are the recent HERA low-x measurements [7], because UHE neutrino cross sections receive a dominant contribution from the ultrasmall-x region of DIS – even below the range covered by HERA.

The total neutrino-nucleon cross section is given by

$$\sigma_{\rm tot}^{\nu N\to X}(s=2M_N E_\nu) = \int_0^1 dx \int_0^{sx} dQ^2\; d^2\sigma^{\nu N\to X}/dxdQ^2\,, \tag{1}$$

with the standard (weak) DIS cross section of the form [1]

$$\frac{d^2\sigma^{\nu N\to X}}{dxdQ^2} = \frac{G_F^2}{\pi}\left(\frac{M_{W,Z}^2}{Q^2+M_{W,Z}^2}\right)^2 [q(x,Q^2)+\bar{q}(x,Q^2)(1-y)^2]\,, \tag{2}$$

where $y = Q^2/(xs)$. The effective quark and anti-quark distributions $q(x,Q^2)$ and $\bar{q}(x,Q^2)$ include the appropriate (electro-)weak couplings for charged or neutral current DIS. In what follows we will always assume an isoscalar nucleus $[N=(p+n)/2]$. Note, that Eq. (1) is well behaved only for neutrino DIS where – contrary to the case of photon exchange – the heavy vector meson propagator in Eq. (2) is non-singular [2] at $Q^2 = xys = 0$. Accordingly, the following arguments are tied to the weak interaction framework, with no analogue for the more familiar photon exchange process. For explicit calculations we restrict ourselves to neutrino, rather than anti-neutrino, scattering, with the understanding that sea-quark dominance in the UHE limit does not discriminate between neutrinos and anti-neutrinos.

Recent evaluations of UHE cross sections have extrapolated PDF's below $x < 10^{-4}$ using power laws [4], radiative renormalization group behaviour [5] or BFKL-type resummations [6]. Within the accuracy required for astrophysical phenomenology these approaches agree and predict a steep rise of the low-x parton density functions. Any flattening of the PDF's at low x due to gluon recombination is naively expected to be absent at the UHE scale $Q \simeq M_W$ due to the higher twist nature $\mathcal{O}(1/Q^2)$ of such recombination channels [15].

To gain a feeling for the interplay between the UHE cross section and the low-x behaviour of the PDF's we can perform the Q^2 integral under the approximation that we neglect the evolution of the PDF's. We find[3]

$$\sigma_{\rm tot}^{\nu N\to X}(s) \simeq \frac{M_{W,Z}^2 G_F^2}{\pi}\int_0^1 dx \left[q(x)\left(\frac{\hat{s}}{1+\hat{s}}\right) + \bar{q}(x)\left(\frac{2}{\hat{s}}+1-2\left(\frac{1+\hat{s}}{\hat{s}^2}\right)\ln(1+\hat{s})\right)\right]\,, \tag{3}$$

where $\hat{s} = sx/M_{W,Z}^2$ and the PDF's are evaluated at some fixed scale (*i.e.* $q(x) = q(x, M_{W,Z}^2)$, $\bar{q}(x) = \bar{q}(x, M_{W,Z}^2)$). From (3) it is straightforward to show that assumed behaviour of the PDF's

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[1]Sub-leading QCD corrections are implicitly included in the evolution of $q(x) = q(x,Q^2), \bar{q}(x,Q^2)$; explicit $\mathcal{O}(\alpha_s^1)$ corrections to Eq. (1) are negligible $\mathcal{O}(1-2\%)$.

[2]*i.e.* imposing a "$Q >$few GeV" cut-off on the integral in (1) changes its value only marginally.

[3]We have verified that the effects of evolution induce corrections of only about 20%.

at $x \to 0$ determines the $E_\nu \to \infty$ asymptotic behaviour of the total cross section [16]: A power-type $x \to 0$ behaviour $xq(x) \propto x^{-\beta}$ [4] of the sea-quark PDF translates into a power-type rise with energy $\sigma_{\rm tot}^{\nu N \to X} \propto E_\nu^\beta$. Renormalization group evolution from a flat low-scale input [5] gives rise to a slower, but still significant, growth with energy. This is easiest to see in the double-leading-logarithm approximation [17], in which the cross section can be seen analytically [16] to rise less fast than any power but faster than a logarithm. Although each of the different approximations of the small-x behaviour gives different quantitative predictions of the $E_\nu \to \infty$ asymptotics, we observe in Fig. 1 that they agree to a good approximation up to energies as high as $E_\nu = 10^{12}$ GeV. In particular, they all rise continuously with energy.

This rise, which is a direct consequence of the $x \to 0$ behaviour of $q(x)$, $\bar{q}(x)$, reflects mainly the HERA measurements of $F_2^\gamma \simeq x \sum e_q^2 (q + \bar{q})(x)$ at small-x. The question is, therefore, up to what energy is the rise of $\sigma_{\rm tot}$ strictly required by these data? The *ep* collider HERA runs at a center of mass energy of $\sqrt{s} \simeq 300\,{\rm GeV}$ corresponding to an equivalent neutrino energy of $E_\nu^{\rm HERA} \simeq 5 \times 10^4$ GeV [18]. However, leaving theoretical extrapolations aside, $\sigma_{\rm tot}$ for $E_\nu > E_\nu^{\rm HERA}$ is not completely unknown because HERA probes the *differential* $d\sigma/dxdQ^2$ in Eq. (2) down to much lower x (corresponding to higher E_ν) than the average $\langle \log_{10} x \rangle \simeq -1.5$ in the *integrated* $\sigma_{\rm tot}(E_\nu^{\rm HERA})$, albeit at a lower scale $Q < M_{W,Z}$ for the γ^*-exchange process. Hence, the maximal E_ν up to which the evaluation of $\sigma_{\rm tot}$ according to Eqs. (1) and (2) is fixed by HERA depends on the extent to which we trust the NLO QCD scale-evolution of DIS structure functions. Recent estimates [19] of NNLO corrections to parton evolution confirm a trustworthy stability of the full singlet evolution at least down to $x \gtrsim 10^{-4}$ for $Q^2 \gtrsim 10$ GeV^2. This latter kinematical range is well covered by HERA and the corresponding data are incorporated in the PDF sets of Refs. [12–14]. Assuming nothing more[4] than the universality of the PDF's and the validity of the RGE within $10^{-4} \lesssim x$, 10 GeV$^2 < Q^2 < M_W^2$, we can then consider $\sigma_{\rm tot}$ – if evaluated according to Eqs. (1) and (2) – as *effectively* covered by HERA data up to neutrino energies where $\sigma_{\rm tot}$ becomes sensitive to $x < 10^{-4}$. This sets in[5] smoothly above $E_\nu \gtrsim 10^6$ GeV, which we indicate by the vertical line in Fig. 1. Note, this is a very conservative estimate of the impact of the HERA data. Were we to trust the PDF-mediated mapping of the HERA measurements onto the neutrino cross-section down to $x \simeq 3 \times 10^{-5}$, and rely on the evolution from $Q^2 = 1.5$ GeV^2 to M_W^2, then the most recent HERA data [20] would cover $\sigma_{\rm tot}$ at $E_\nu \simeq 10^8$ GeV except for a correction of $\sim 25\%$ from $x < 3 \times 10^{-5}$. As we will deduce in the next section, this would suggest HERA γ-exchange data are already probing a perturbative unitarity bound in UHE

[4]I.e. assumptions which are intrinsic to the perturbative evaluation in Eq. (1).

[5]For illustration, see Fig. 3 in [3] or Fig. 3 in [5].

neutrino scattering. However, for now, we prefer not to speculate about shifting the vertical line in Fig. 1 to the right. A detailed statistical analysis, though, might do better and provide stringent correlations between HERA data and $\sigma_{\rm tot}$ at even higher E_ν. Note, our limitation $x > 10^{-4}$ is also a safe condition for the absence of recombination corrections [15] at HERA scales which are, accordingly, even more negligible at the UHE scale $Q = M_{W,Z}$.

III. PERTURBATIVE UNITARITY

Using unitarity of the S-matrix, we can relate the total νN cross section in (1) to the imaginary part of the neutrino-nucleon forward scattering amplitude:

$$\sqrt{\lambda}\sigma_{\rm tot}^{\nu N\to X}(s) = {\rm Im}\left[T_{\{\nu N,\nu N\}}(s,t=0)\right] , \tag{4}$$

where $\lambda = (s - M_N^2)^2$ and s, t are the standard Mandelstam variables. The elastic amplitude T is related to the elastic cross section

$$\frac{d\sigma_{\rm el}^{\nu N\to\nu N}}{dt} = \frac{1}{16\pi\lambda}\left|T_{\{\nu N,\nu N\}}(s,t)\right|^2 . \tag{5}$$

Combining Eqs. (4) and (5) gives a general limit on $\sigma_{\rm tot}$ in Eq. (1)

$$\left.\frac{d\sigma_{\rm el}}{dt}\right|_{t=0} = \frac{1}{16\pi}\left[\frac{\left({\rm Re}[T_{\{\nu N,\nu N\}}(s,0)]\right)^2}{\lambda} + \sigma_{\rm tot}^2\right] \geq \frac{1}{16\pi}\sigma_{\rm tot}^2 \quad . \tag{6}$$

We note that the inequality Eq. (6), as derived, holds strictly for each spin and isospin state of the nucleon; however, it is straightforward to show that[6]

$$\frac{1}{4}\sum\sigma_{\rm tot}^2 \geq \left[\frac{1}{4}\sum\sigma_{\rm tot}\right]^2 , \tag{7}$$

where the sum is over spin and isospin of the nucleon. Thus, we can just as well use the spin/isospin averaged cross sections in Eq. (6)[7].

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[6]The difference between the left and right hand sides of Eq. (7) is the variance of elementary probability theory and therefore positive.

[7]In fact, due to constraints on spin and isospin asymmetries in DIS scattering, one would expect the difference between the right-hand and left-hand sides of Eq. (7) to be negligible in the high energy limit.

The inequality in Eq. (6) is a standard statement of the optical theorem and can be found in many textbooks. It follows strictly from the positivity of $(\mathrm{Re}T)^2$ and does not rely on any perturbative expansion. We now consider its implications in the context of a perturbative calculation in the weak coupling. Expanding the elastic amplitude to lowest power in the weak coupling g and using the most general Z-nucleon coupling (see Fig. 2), we obtain

$$\begin{aligned} T = & \left(\frac{g}{4\cos\theta_W}\right)^2 \bar{u}_\nu(k')\,\gamma^\alpha\,(1-\gamma_5)\,u_\nu(k)\,\frac{1}{t-M_Z^2} \\ & \times\,\bar{u}_N(p')\,\Big[\gamma_\alpha\,f_1(q^2) + i\sigma_{\alpha\beta}\,q^\beta\,f_2(q^2) + q_\alpha\,f_3(q^2) + \gamma_\alpha\,\gamma_5\,g_1(q^2) \\ & + i\sigma_{\alpha\beta}\,\gamma_5\,q^\beta\,g_2(q^2) + q_\alpha\,\gamma_5\,g_3(q^2)\Big]\,u_N(p) + \mathcal{O}(g^4)\,. \end{aligned} \tag{8}$$

In the forward direction only the form factors $f_1(0)$ and $g_1(0)$ contribute. Using the standard model Z-boson current and isospin symmetry, the values of the form factors at $q^2 = 0$ can be expressed in terms of the nucleon isospin operator T_3 and charge operator Q as $f_1(0) = (2\,T_3 - 4\sin^2\theta_W\,Q)$ and $g_1(0) = -2\,T_3\,g_A$, where $g_A = 1.27$ is the measured value [21] of the axial vector coupling constant. For an isoscalar nucleus we obtain

$$\left.\frac{d\sigma_{\mathrm{el}}^{(2)}}{dt}\right|_{t=0} = \frac{G_F^2}{8\pi}\left[\frac{1+(1-4\sin^2\theta_W)^2}{2} + g_A^2\right]\,. \tag{9}$$

Accordingly, the inequality Eq. (6) gives

$$\sigma_{\mathrm{tot}} \lesssim 9.3\times 10^{-33}\ \mathrm{cm}^2\,. \tag{10}$$

The value on the right-hand side of Eq. (10) is shown as a horizontal line in Fig. 1. We observe that the inequality is violated for $E_\nu \gtrsim 2\times 10^8$ GeV. Note that the precise value of the bound is rather insensitive to the values of $f_1(0)$ and $g_1(0)$ because it depends on the square root of the factor in the square brackets of Eq. (9).

HERA information on PDF's determines σ_{tot} for neutrino energies as high as $E_\nu \gtrsim 10^6$ GeV. Perhaps large changes in the PDF's will set in within the range of x appropriate for $E_\nu \sim 10^6 - 10^8$ GeV such that the bound is respected for larger E_ν. These corrections would presumably arise as a softening of the gluon and sea quark distributions at small x.

Alternatively, one can note that the bound Eq. (10) is obtained by comparing an $\mathcal{O}(g^4)$ expression for σ_{tot} with an $\mathcal{O}(g^2)$ expression for the forward elastic scattering amplitude. Since unitarity is an order-by-order statement within perturbation theory, to be strictly rigorous we must also include the absorptive $\mathcal{O}(g^4)$ contribution to the forward scattering amplitude. In this way, unitarity in the ultrahigh energy limit can in principle be restored, but at the cost of large $\mathcal{O}(g^4)$ corrections to the forward amplitude.

For illustration, we write the elastic amplitude symbolically as

$$T = g^2 T^{(2)} + g^4 T^{(4)} + \ldots . \tag{11}$$

where we have calculated the $T^{(2)}$ part above and the $T^{(4)}$ term results from the exchange of a pair of virtual Z^0's or $W^{\pm}$'s in the elastic scattering, as illustrated in Fig. 3. Using this notation

$$\left.\frac{d\sigma_{\rm el}}{dt}\right|_{t=0} \propto \; g^4 {T^{(2)}}^2 + 2g^6 T^{(2)} \; \mathrm{Re}[T^{(4)}] + g^8 \left[\left(\mathrm{Re}[T^{(4)}]\right)^2 + \left(\mathrm{Im}[T^{(4)}]\right)^2 + \ldots\right] + \ldots , \tag{12}$$

and

$$(\sigma_{\rm tot})^2 \propto g^8 \left(\mathrm{Im}[T^{(4)}]\right)^2 + \ldots . \tag{13}$$

From these equations we see that the unitarity condition Eq. (6) can be satisfied; however it requires that the *higher order* $g^4T^{(4)}$ term become larger than the *lower order* $g^2T^{(2)}$ term in the high energy limit. So, in this approach, calculating to lowest order in the weak coupling g is not a good approximation for the elastic differential cross section at high energy. But if the perturbation expansion is unreliable in Eq. (12), then it is also reasonable to expect higher order corrections to $\mathrm{Im}\, T$ to be larger than the term shown, both in Eq. (12) and in Eq. (13). Since this is the term on which all calculations have been based it implies that these calculations may not be reliable at very high energies. If the perturbation expansion doesn't work in Eq. (12) then we must question its validity in Eq. (13).

IV. CONCLUSIONS

We compared perturbative predictions for neutrino-nucleon total cross sections at ultrahigh neutrino energies to a unitarity bound derived from the corresponding elastic neutrino scattering amplitude in the forward direction. At the non-perturbative level the bound is absolutely stringent. A lowest order expansion in powers of the weak coupling leads to a bound which is saturated at an energy surprisingly close to what is effectively covered by HERA measurements. Thus, for the largest energies relevant to neutrino astronomy, sizable PDF corrections could reside in $\sigma_{\rm tot}$. Alternatively, the large apparent $\mathcal{O}(g^4)$ total cross section should manifest itself through the corresponding term in the elastic amplitude as well. But if the perturbation expansion is not valid for the elastic amplitude how can we trust it for the inelastic cross section? In this case also $\sigma_{\rm tot}$ could be very different for large E_ν than the curves shown in Fig. 1.

We have ignored the possibility that unitarity is satisfied by a new interaction – physics beyond the standard model [8–11]. But even in this alternative our basic conclusions remain the same: something dramatic must happen at energies close to the current energy and the existing calculations of $\sigma_{\rm tot}$ should not be trusted at large neutrino energy.

If the bound is correct, and the PDF's change at low x, then $\sigma_{\rm tot}$ will be smaller at high E_ν than what is shown in Fig. 1. If perturbation theory has broken down then $\sigma_{\rm tot}$ could be anything - perhaps even large enough to explain the cosmic ray data without the need for new physics. Or perhaps the values shown in Fig. 1 could turn out to be correct; maybe weak perturbation theory only breaks down for the elastic cross section. Without a better knowledge of the PDF's at low x, there is no way to know.

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FIGURES

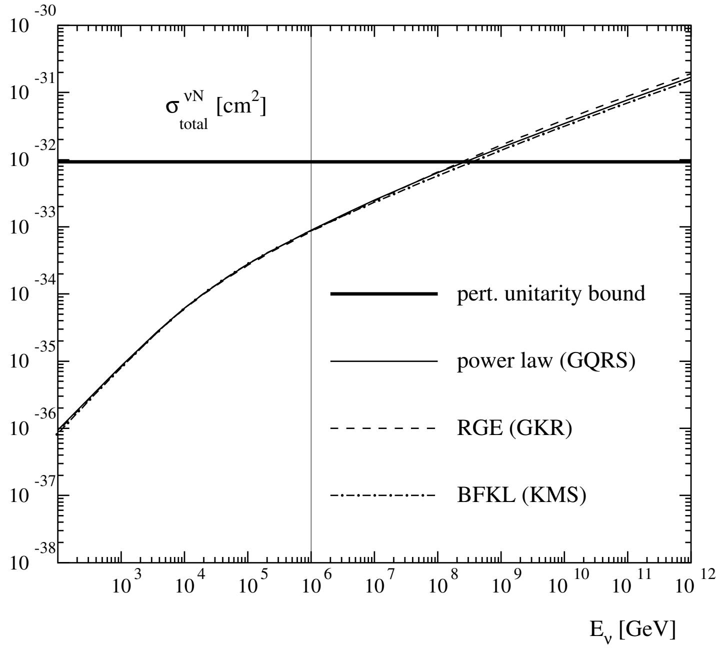

Fig. 1

FIG. 1. Perturbatively evaluated QCD neutrino-nucleon total cross sections from a power law (GQRS)[4], renormalization group evolution (GKR)[5], or BFKL (KMS)[6] approach towards ultrasmall-x structure functions. The thick horizontal line is the lowest order perturbative unitarity bound in Eq. (10). The vertical line is a conservative upper bound on the E_ν-range effectively covered by HERA, *i.e.* the perturbative cross section for energies $E_\nu \lesssim 10^6$ GeV receives no significant contribution from the $x \lesssim 10^{-4}$ regime of DIS.

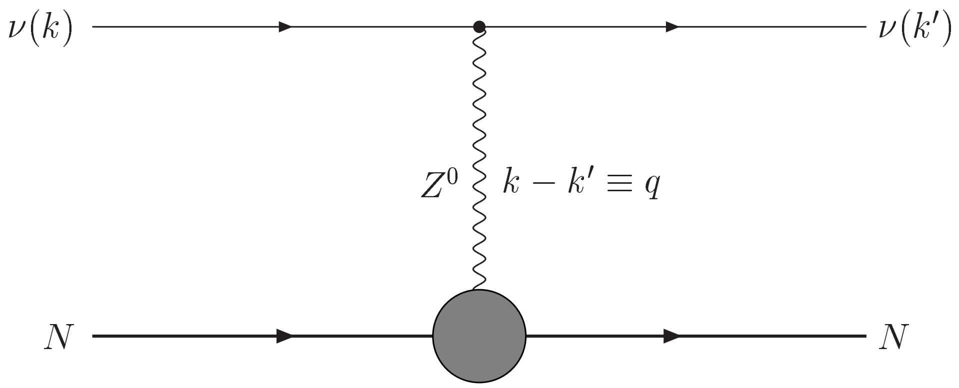

FIG. 2. Lowest order $[\mathcal{O}(g^2)]$ elastic amplitude $T_{\{\nu N,\nu N\}}$ corresponding to Eq. (8).

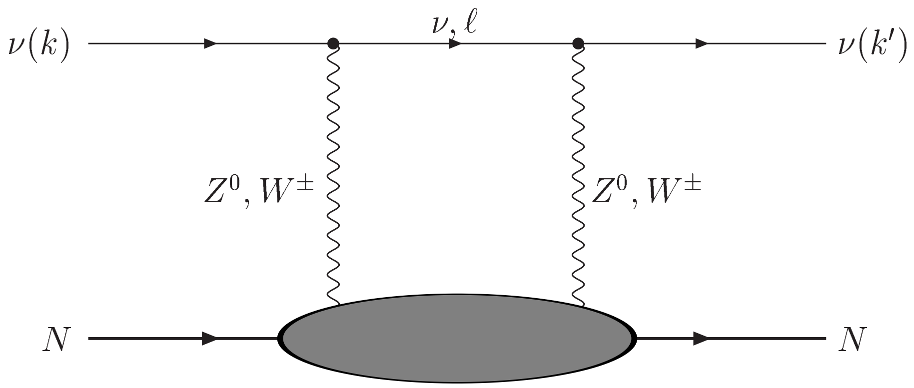

FIG. 3. Next order $[\mathcal{O}(g^4)]$ elastic Amplitude $T_{\{\nu N, \nu N\}}$ corresponding to $T^{(4)}$ in Eq. (11).