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A Monte Carlo Solution to the Balitsky-Fadin-Kuraev-Lipatov Equation for Resummation in Perturbative QCD

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Abstract

A simple solution to the BFKL equation is obtained as a series in the number of real gluons emitted with transverse momentum greater than some small cutoff μ. This solution reveals physics inside the BFKL ladder which is hidden in the standard inclusive solution, and lends itself to a straightforward Monte Carlo implementation. With this approach one can explore new useful physical observables, which are shown to be independent of the cutoff μ. In addition, this approach allows the imposition of kinematic constraints (such as energy conservation) which are important at finite energies. The distribution of $\sum E_\perp$ of particles in a central rapidity bin between two widely-spaced jets is presented as an example.

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) equation [1], which systematically resums powers of α_s times large rapidity intervals or logarithms of Feynman x in perturbative QCD, has recently moved from the purely theoretical realm to the phenomenological arena. The classic prediction of the BFKL resummation is the rise of $F_2(x)$ at small x in deep inelastic scattering. Unfortunately, due to the resiliency of the parton density functions and DGLAP [2] evolution equations, this observation [3, 4] is still open to interpretation [5]. Alternatively, by tagging jets on both ends of a large rapidity interval, as suggested by Mueller and Navalet [6], it is possible to unambiguously isolate the effects of the BFKL ladder from the parton density functions. Predictions of this kind include the rise in the cross section as a function of the rapidity interval, both in hadron-hadron [6] and lepton-hadron colliders [7]. In particular, recent preliminary results from H1 [3] on deep inelastic scattering with a tagged jet at fixed x_j show an intriguing rise in the cross section with decreasing x_{BJ} that is not well-explained either by a fixed-order matrix element calculation [8] or by parton shower Monte Carlo simulations. The BFKL prediction [9], however, seems to be in good agreement. Other observables related to the kinematics of the tagging jets, such as the decorrelation in azimuthal angle [10], have also been considered.

With the advent of BFKL phenomenology, it has become a necessity to understand the range of validity and the errors inherent in the BFKL approximations. The calculation of the next-to-leading logarithmic (NLL) corrections to the BFKL matrix elements is currently under way [11]. Short of these full NLL corrections, it has been shown in [12] that some of the largest corrections to the asymptotic theory are purely kinematic in origin. In the standard solution to the BFKL equation the transverse momenta of the ladder gluons are integrated from zero to infinity. However, at physical energies the inclusion of kinematic constraints on these integrals can significantly modify the predictions of the theory, even if the difference is formally subleading in the asymptotic expansion. In Ref. [12] an effective rapidity interval was defined as an attempt to include some of these kinematic effects in the BFKL calculation.

In this paper we present a new solution to the BFKL equation in a form naturally suited for physical interpretation and Monte Carlo implementation [13, 14]. This approach also offers a natural way to impose kinematic constraints and to assess the uncertainties due to the asymptotic nature of the equation. The key ingredient to this solution is the introduction of a lower cutoff μ on the transverse momentum of the real gluons that are produced. However, the cutoff is introduced in such a way that, for sufficiently small μ, any infrared safe observable is independent of the cutoff. In the inclusive case the Monte Carlo method generates an exact answer, identical to the known solution in the literature, with no arbitrary parameters. In addition, new experimental observables which depend on the internal structure of the BFKL ladder can be obtained [15].

Let us begin with a general description of a semi-hard process, where the parton-parton center-of-momentum energy $\sqrt{\hat{s}}$ is much larger than the typical momentum transfer scale Q. In this limit the partonic cross section factorizes into the form

$$\frac{d\hat{\sigma}}{d^2\boldsymbol{p_{a\perp}}d^2\boldsymbol{p_{b\perp}}} = V_a(p_{a\perp}^2)\, f(y_{ab},\boldsymbol{p_{a\perp}},\boldsymbol{p_{b\perp}})\, V_b(p_{b\perp}^2)\ . \tag{1}$$

A physical interpretation of this form of the cross section is represented in Fig. 1. The process consists of two distinct scatterings, which occur at widely-separated rapidities, y_a and y_b, and small transverse momenta $p_{a\perp} = |\boldsymbol{p_{a\perp}}| \sim p_{b\perp} = |\boldsymbol{p_{b\perp}}| \sim Q$. Each scattering form factor, $V_a(p_{a\perp})$ and $V_b(p_{b\perp})$, depends only on the transverse momentum which flows into its particular vertex. The precise form of the form factors, however, depends on the specific partons involved in the scatterings. The function $f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}})$, which connects the two scatterings, is essentially a propagator which allows $\boldsymbol{p_{b\perp}}$ to flow to $\boldsymbol{p_{a\perp}}$ by emitting gluons over a rapidity interval $y_{ab} = y_a - y_b \sim \ln(\hat{s}/p_{a\perp}p_{b\perp})$. This function is universal, and it is our object of interest.

The form of the cross section given in (1) is valid in the limit of large y_{ab}. In that limit the perturbative contributions to $f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}})$ that are leading in $\alpha_s y_{ab}$ can be resummed systematically with the aid of the BFKL equation. We now present the equation and describe the physics that it encodes. It can be written

$$\begin{aligned}\frac{\partial f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}})}{\partial y_a} &= \frac{\bar{\alpha}_s}{\pi} \int \frac{d^2\boldsymbol{k}_\perp}{k_\perp^2} \Bigg[f(y_{ab}, \boldsymbol{p_{a\perp}} + \boldsymbol{k}_\perp, \boldsymbol{p_{b\perp}}) \\ &\qquad - \frac{p_{a\perp}^2}{k_\perp^2 + (\boldsymbol{p_{a\perp}} + \boldsymbol{k}_\perp)^2} f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) \Bigg] ,\end{aligned} \tag{2}$$

where $\bar{\alpha}_s = \alpha_s N_c/\pi$. The boundary condition for the equation is

$$f(0, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) = \frac{1}{2}\delta^{(2)}(\boldsymbol{p_{a\perp}} + \boldsymbol{p_{b\perp}}) , \tag{3}$$

which corresponds to no gluon emissions and enforces conservation of transverse momentum. The first term on the right-hand side of the BFKL equation (2), upon iterating from the boundary condition n times, gives the squared amplitude for producing n real gluons, in the approximation that the gluons are well-separated in rapidity. The amplitude with three gluon emissions is represented by the Feynman diagram in Fig. 2. The second term on the right hand side of (2) gives the virtual corrections which "reggeize" the t-channel gluon propagators, represented by the heavy solid lines in the figure.

An important point here is that the singularities in the real and virtual terms of the integrand cancel as $k_\perp$ becomes small. Thus, we can cut off the integral at $k_\perp = \mu$ for both the real and virtual gluons, and the neglected contribution will vanish as $\mu \to 0$. With this cutoff we can explicitly do the integration over the virtual gluon corrections, obtaining:

$$\begin{aligned} \frac{\partial f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}})}{\partial y_a} &= \frac{\bar{\alpha}_s}{\pi} \int \frac{d^2\boldsymbol{k}_\perp}{k_\perp^2} f(y_{ab}, \boldsymbol{p_{a\perp}}+\boldsymbol{k}_\perp, \boldsymbol{p_{b\perp}}) \\ &\quad + \bar{\alpha}_s \ln(\mu^2/p_{a\perp}^2) f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) \;+\; \mathcal{O}(\mu^2/p_{a\perp}^2) \;, \end{aligned} \tag{4}$$

where the integral over real gluons is restricted to $k_\perp > \mu$. From hereon we neglect the terms of $\mathcal{O}(\mu^2/p_{a\perp}^2)$. Then we can simplify this equation further by making the substitution

$$f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) \;=\; \left(\frac{\mu^2}{p_{a\perp}^2}\right)^{\bar{\alpha}_s y_{ab}} \tilde{f}(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) \;, \tag{5}$$

which leaves the following equation [16]:

$$\frac{\partial \tilde{f}(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}})}{\partial y_a} \;=\; \frac{\bar{\alpha}_s}{\pi} \int \frac{d^2\boldsymbol{k}_\perp}{k_\perp^2} \left(\frac{p_{a\perp}^2}{(\boldsymbol{p_{a\perp}}+\boldsymbol{k}_\perp)^2}\right)^{\bar{\alpha}_s y_{ab}} \tilde{f}(y_{ab}, \boldsymbol{p_{a\perp}}+\boldsymbol{k}_\perp, \boldsymbol{p_{b\perp}}) \;, \tag{6}$$

with the same boundary condition (3).

In (5) we see that the dependence of the virtual corrections on μ simply exponentiates into an overall factor. Starting with the boundary condition, we can iterate equation (6) to obtain a series solution $f = \sum_{n=0}^{\infty} f^n$, where the nth term is the contribution from the emission of n real gluons having $k_\perp > \mu$. This contribution can be written as a product of integrals over the phase space of each gluon in the ladder:

$$f^n(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) = \int \prod_{i=1}^{n} \left[\bar{\alpha}_s dy_i \frac{dk_{i\perp}^2}{k_{i\perp}^2} \frac{d\phi_i}{2\pi} \left(\frac{\mu^2}{q_{i\perp}^2} \right)^{\bar{\alpha}_s y_{i+1,i}} \right] \left(\frac{\mu^2}{q_{0\perp}^2} \right)^{\bar{\alpha}_s y_{1,0}} \times \frac{1}{2} \delta^{(2)}(\boldsymbol{p_{a\perp}} + \boldsymbol{q_{n\perp}}) \tag{7}$$

with

$$\boldsymbol{q_{j\perp}} = \boldsymbol{p_{b\perp}} + \sum_{i=1}^{j} \boldsymbol{k_{i\perp}} \tag{8}$$

and $\boldsymbol{q_{0\perp}} = \boldsymbol{p_{b\perp}}$. The integrals in rapidity are ordered with $y_b \equiv y_0 < y_1 < \cdots < y_n < y_{n+1} \equiv y_a$. Written in this manner, the solution recovers the simple interpretation as the emission of real gluons by the exchange of "reggeized" gluons in the t-channel as in Fig. 2, with μ the infrared cutoff to the gluon Regge trajectory. For clarity, we emphasize that only the gluons exchanged in the t-channel are "reggeized", while the emitted partons are standard gluons.

Each term in the series is positive definite. Therefore, it is straightforward to implement this solution as a Monte Carlo simulation. Given $\boldsymbol{p_{b\perp}}$ and the rapidity interval $y_a - y_b$, we first sample the distribution in the number n of gluons in the ladder. Next, we produce the four-momenta of the n gluons successively as given by the distribution (7). Finally, we fix $\boldsymbol{p_{a\perp}}$ by conservation of transverse momentum. In practice the events are produced using approximate distributions and are then reweighted.

To understand more clearly the cutoff dependence of this solution, let us calculate the quantity $F = \int f \, dp_{a\perp}^2 d\phi_a$, where the integral just fixes $\boldsymbol{p_{a\perp}}$ via the δ-function in (7). For simplicity of the discussion here, it is convenient to consider a modified equation, obtained by replacing $q_{i\perp} \to p_{b\perp}$ everywhere in (7) and by setting the upper limit on the integrations to $k_\perp^2 = p_{b\perp}^2$. Now the nested integrals can be done analytically, the series can be summed, and the dependence on μ vanishes identically. In fact we obtain $F = 1$, and the distribution in the number of ladder gluons is just a Poisson distribution with mean

$$\langle n \rangle \;=\; \bar{\alpha}_s y_{ab} \ln\left(p_{b\perp}^2/\mu^2\right) \; . \tag{9}$$

The physical significance of the cutoff is now apparent. As we lower μ, the number of gluons emitted in the rapidity interval grows logarithmically. However, the average $k_\perp$ of each gluon is reduced in such a way that, for suitable infrared-finite observables, the dependence on the cutoff vanishes. In the exact solution (7) the distribution in the number of ladder gluons will differ somewhat, but the qualitative features of this discussion still apply.

We now present in Fig. 3 a plot of the Monte Carlo solution compared to the standard BFKL solution [1],

$$f(y_{ab}, \boldsymbol{p_{a\perp}}, \boldsymbol{p_{b\perp}}) \;=\; \frac{1}{(2\pi)^2 p_{a\perp} p_{b\perp}} \sum_{n=-\infty}^{\infty} e^{in(\phi_{ab}-\pi)} \int_{-\infty}^{\infty} d\nu \, e^{\omega(n,\nu)\, y_{ab}} \left(p_{a\perp}^2/p_{b\perp}^2\right)^{i\nu} \; , \tag{10}$$

with $\phi_{ab} = \phi_a - \phi_b$ and

$$\omega(n,\nu) \;=\; 2\bar{\alpha}_s[\psi(1) - \mathrm{Re}\,\psi(\frac{|n|+1}{2} + i\nu)] \; , \tag{11}$$

where ψ is the logarithmic derivative of the Gamma function. In this figure we fix $p_{b\perp} = 50$ GeV and $y_{ab} = 4$, and we plot $\int f \, d\phi$ as a function of $p_{a\perp}$. The Monte Carlo solution agrees with the standard solution (10) for this curve as long as μ is smaller than the bin size used near the peak.

Next, we consider the application to observables that depend on the internal structure of the BFKL ladder. As discussed earlier we are interested in observables that are suitably infrared finite, so that there is no dependence on the cutoff scale μ. More exclusive observables, such as parton or jet multiplicity distributions, would require the more sophisticated treatment of infrared singularities at small x given by Catani, Ciafaloni, Fiorani, and Marchesini (CCFM) [17, 18], as well time-like showering off the ladder gluons [19]. For the present analysis we avoid the clustering of partons into jets and concentrate on variables such as transverse energy flow within the rapidity interval. As an example, we calculate such an observable for the Tevatron at $\sqrt{s} = 1800$ GeV. We tag on the two jets a and b with the largest and smallest rapidities with $p_{a\perp}, p_{b\perp} > 20$ GeV. Then we sum the $E_\perp$ of all the particles within a bin in rapidity and plot the distribution in $\Sigma_\perp = \sum E_\perp$. In Fig. 4 we plot the distribution of $\Sigma_\perp$ summed in the bin $|y| < 0.5$ for fixed values of $y_a = 2.5$ and $y_b = -2.5$ for the tagging jets. The distribution is independent of the cutoff μ as long as it is smaller than the bin size. We have checked this by varying μ over many orders of magnitude with no effect on the plot. In this calculation we have also made an improvement to the BFKL prediction by including the kinematic contribution of all of the physically-produced particles to the Feynman-x parameters in the parton density functions of the proton and anti-proton. In practice, this makes a large effect due to the constraint of total energy conservation imposed by the parton density functions. More details of this kinematic improvement and other phenomenological results will be reported in an expanded paper [20].

In conclusion, we have presented a solution to the BFKL equation for large rapidity intervals as a series in the number of real gluons emitted above a cutoff μ. We have incorporated this solution into a Monte Carlo simulation, and we have shown that it reproduces exactly the BFKL dynamics with no dependence on the cutoff for small μ. An advantage of a Monte Carlo solution to BFKL is that it allows one to study the effects of the gluons emitted in the middle of the ladder and even to make experimental cuts on these ladder gluons. Finally, with the Monte Carlo simulation it is also possible to improve the convergence of the resummation by including kinematical effects exactly in the cross section.

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iteratively in the number of "resolved" gluons. However, in contrast to the present paper, the cutoff μ was treated as a physical parameter and so the contributions from $k_\perp < \mu$ had to be retained. Here, treating μ as a small artificial cutoff, the analysis simplifies greatly.

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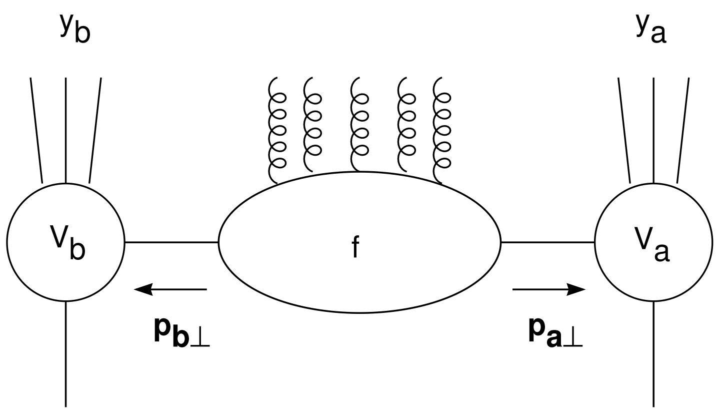

Fig. 1: A schematic picture of the cross section for producing particles at large rapidity separation.

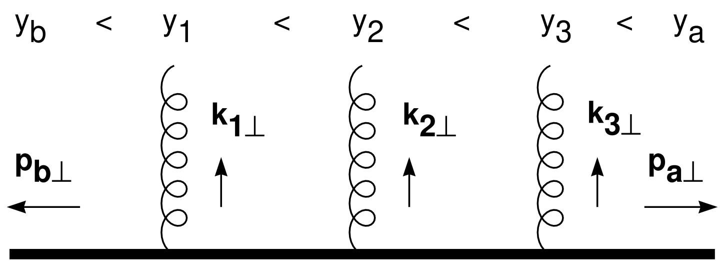

Fig. 2: The amplitude for three gluon emissions in the BFKL approximation.

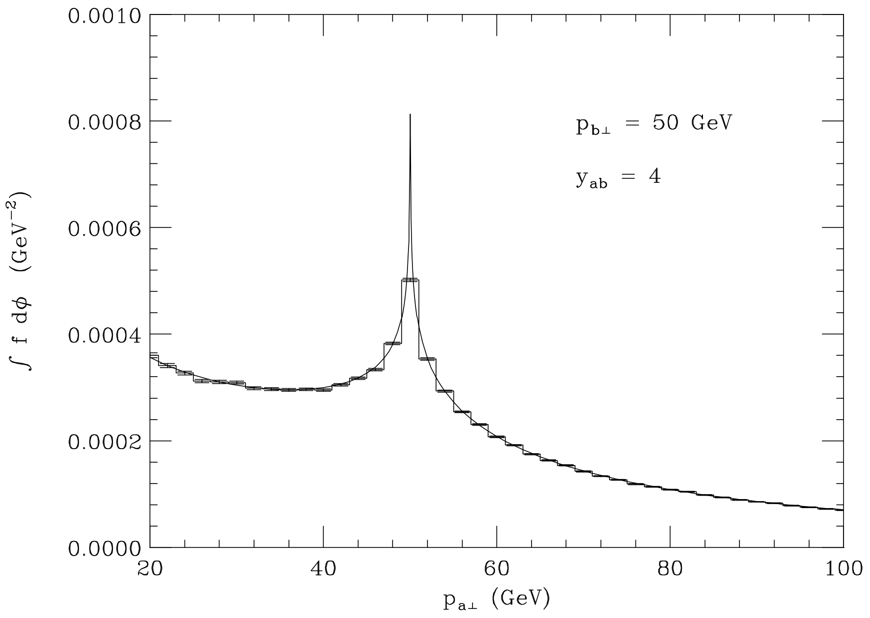

Fig. 3: Comparison of the Monte Carlo solution (histogram) with the standard solution to BFKL (solid).

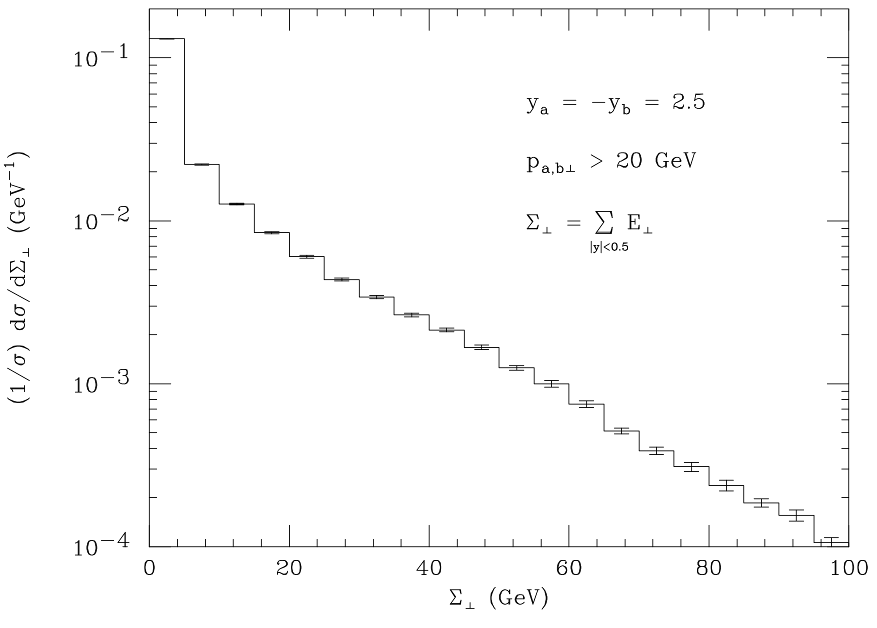

Fig. 4: The distribution of $\Sigma_\perp$ as explained in the text.