Oscillations and Electromagnetic Waves

Three Polarizers

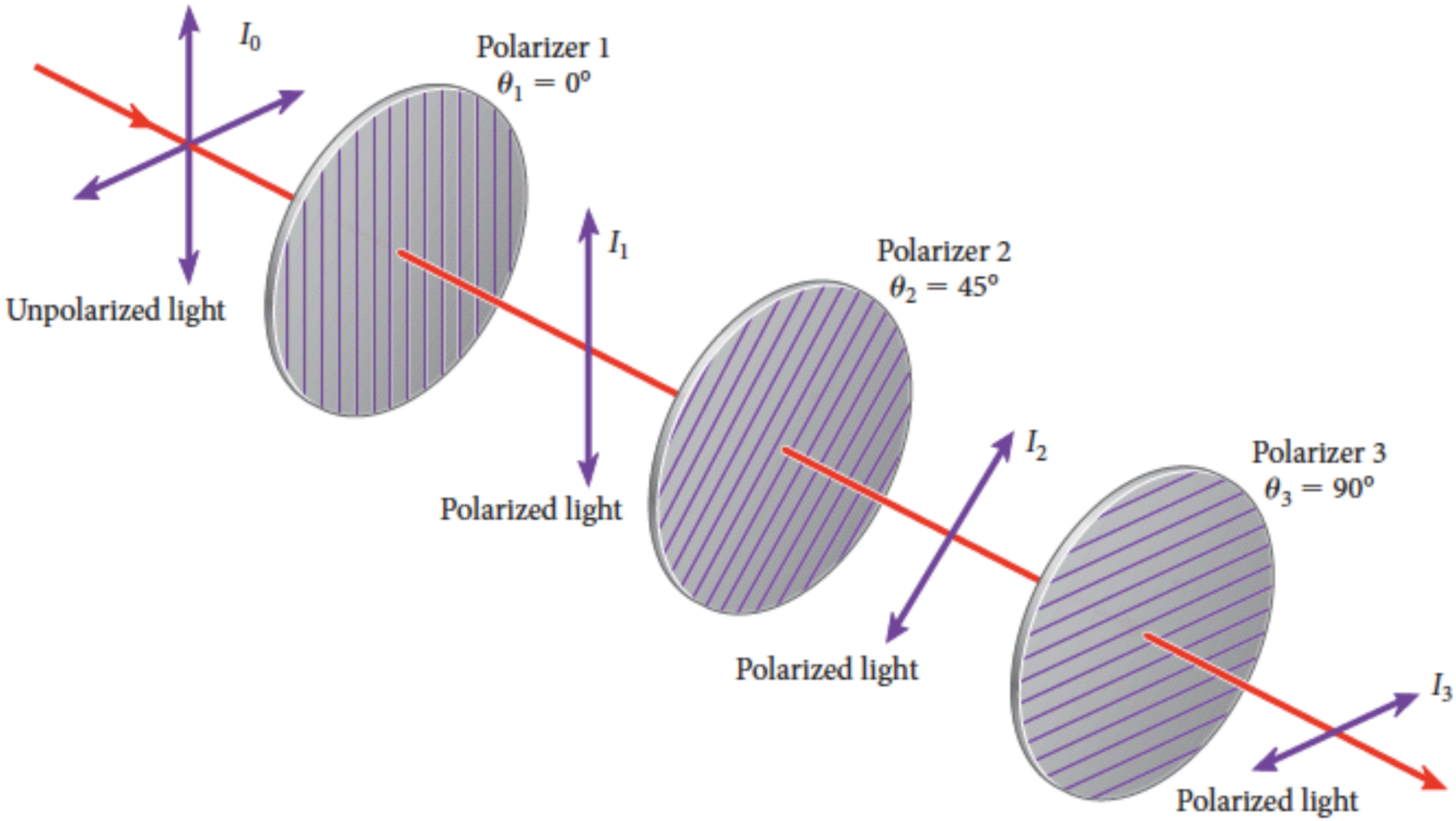
- Consider the case of unpolarized light with intensity I_0 incident on three polarizers
- The first polarizer has a polarizing direction that is vertical, $\theta_1 = 0°$
- The second polarizer has a polarizing angle of $\theta_2 = 45°$ with respect to the vertical
- The third polarizer has a polarizing angle of $\theta_3 = 90°$ with respect to the vertical

PROBLEM

- What is the intensity of the light passing through all the polarizers in terms of the initial intensity?

Three Polarizers

SOLUTION



Three Polarizers

- The intensity of the unpolarized light is I_0
- The intensity of the light passing the first polarizer is

$$I_1 = \frac{1}{2} I_0$$

- The intensity of the light passing the second polarizer is

$$I_2 = I_1 \cos^2(45° - 0°) = I_1 \cos^2(45°) = \frac{1}{2} I_0 \cos^2(45°)$$

- The intensity of the light passing the third polarizer is

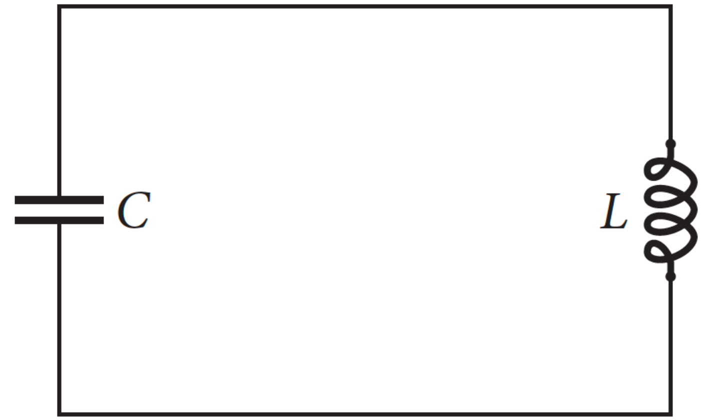
$$I_3 = I_2 \cos^2(90° - 45°) = I_2 \cos^2(45°) = \frac{1}{2} I_0 \cos^4(45°) = I_0 / 8$$

- The fact that 1/8th of the intensity of the light is transmitted is somewhat surprising because polarizers 1 and 3 have polarizing angles that are perpendicular to each other

LC circuit

- Energy:

$$U = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$



- Sinusoidal oscillation

$$q = q_{\max}\cos(\omega_0 t - \phi) \quad (\text{Note the } - \text{ sign})$$

$$i = -i_{\max}\sin(\omega_0 t - \phi)$$

- With $i_{\max} = \omega_0 q_{\max}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

LC Clicker

- The capacitor in a LC circuit is initially uncharged, but there is a current I_0 flowing through the inductor. Will the circuit oscillate?

A. No, because the initial voltage is zero.

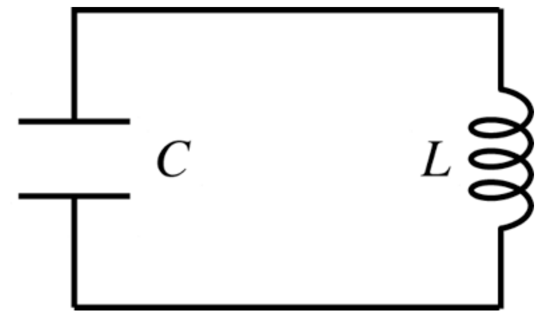
B. No, because the current is only in the inductor

C. Yes, because the initial voltage is not zero.

D. Yes, because the initial current will decrease which results in an induced voltage.

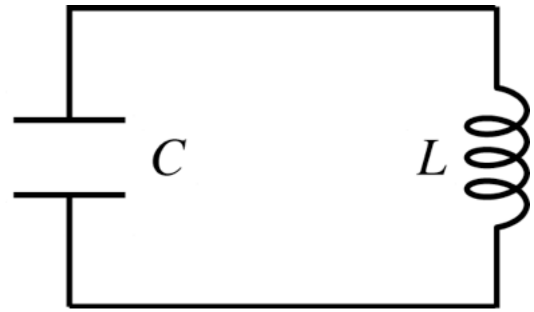
E. None of the above.

LC circuit



- An oscillating LC circuit consists of a 75mH inductor and a 3.60 μF capacitor. The maximum charge on the capacitor is 2.9 μC. What is the maximum current?
- Key ideas: calculate the total energy $U=U_E$ contained in the circuit from what you know about the capacitor. Then use energy conservation and $U_B=0.5i^2L$

LC circuit



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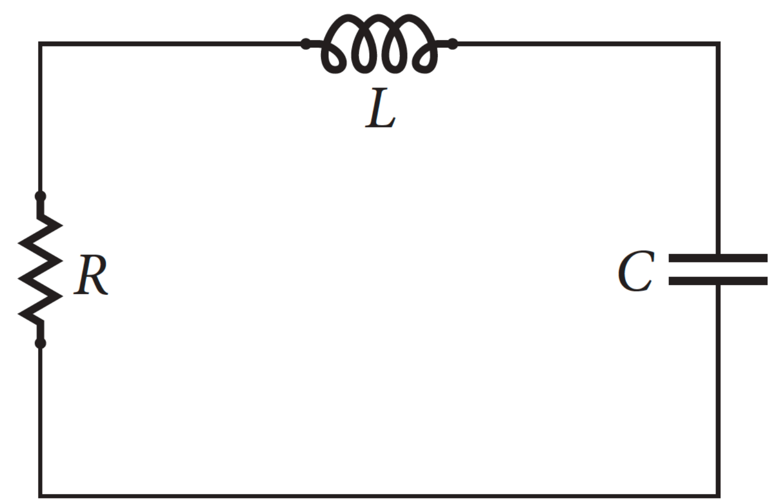
1. All the energy in the circuit resides in the capacitor when it has its maximum charge. The current is then 0.

$$U = \frac{q_{max}^2}{2C} = 1.17 \cdot 10^{-6} J$$

2. 2. When the capacitor is fully discharged, the current is maximum and the total energy resides in the inductor:

$$i = \sqrt{\frac{2U}{L}} = \sqrt{\frac{2(1.17 \cdot 10^{-6} J)}{75 \cdot 10^{-3} H}} = 5.59 \cdot 10^{-3} A$$

RLC circuit



- Amplitude

$$q = q_{max} e^{-\frac{Rt}{2L}} \cos(\omega t)$$

- Frequency

$$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- Energy

$$U_E = \frac{q_{max}^2}{2C} e^{-\frac{Rt}{L}} \cos^2(\omega t)$$

Driven RLC Circuit

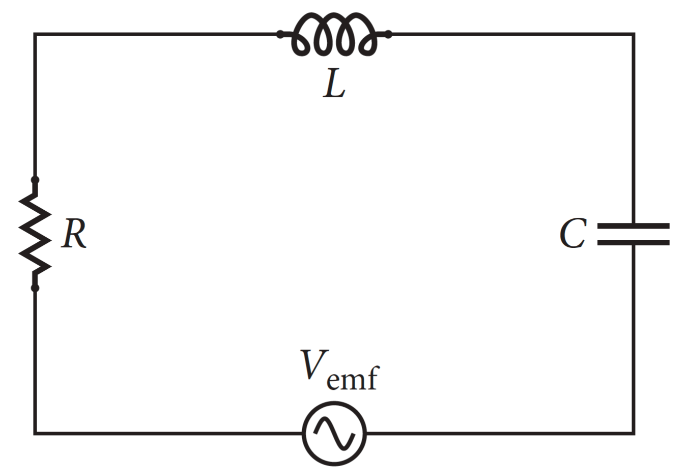
- The driving voltage in a time-varying RLC circuit is given by

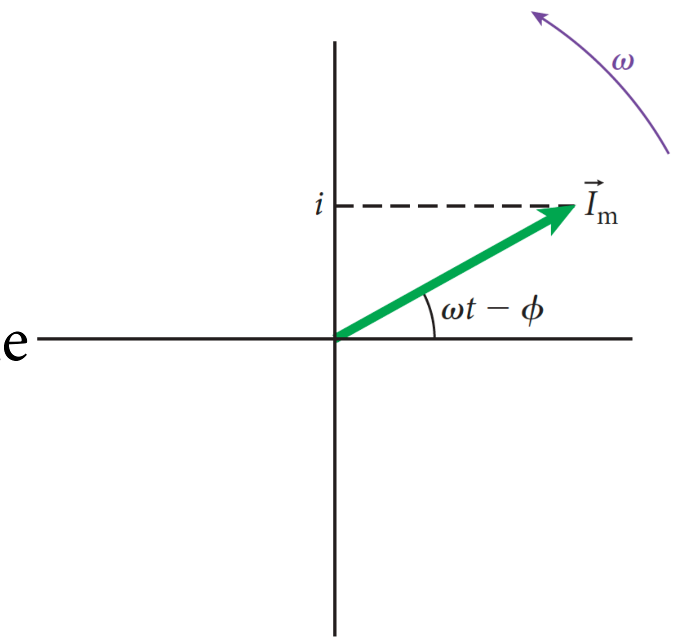
$$V_{emf} = V_{max} \sin \omega t$$

- We describe the time-varying current in driven RLC circuit elements using a phasor I_m

$$i = I_m \sin(\omega t - \phi)$$

- The current is in phase with the voltage drop across the resistor
- We describe the voltage in terms of a phasor V_R
- The time-varying currents and voltages in the circuit can have different phases





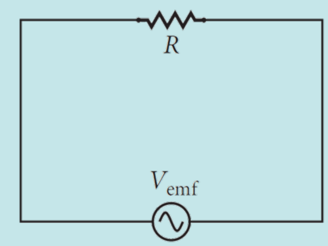
Resistance and Reactance

- Time-varying emf

$$V_{emf} = V_{max} \sin \omega t$$

$$i_R = \frac{V_R}{R} = I_R \sin \omega t$$

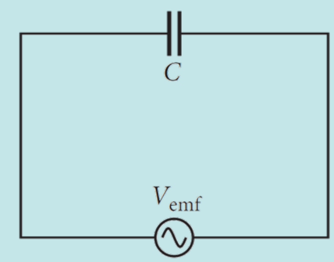
Resistance R



$$X_C = \frac{1}{\omega C}$$

$$i_C = \frac{V_C}{X_C} \sin(\omega t + 90°)$$

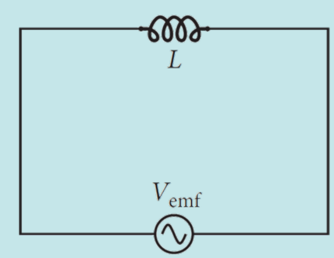
Capacitive Reactance X_C



$$X_L = \omega L$$

$$i_L = \frac{V_L}{X_L} \sin(\omega t - 90°)$$

Inductive Reactance X_L



Driven RLC Circuit

- The sum of the two phasors V_L - V_C and V_R must equal V_m

$$V_m^2 = V_R^2 + (V_L - V_C)^2$$

- The impedance Z is given by

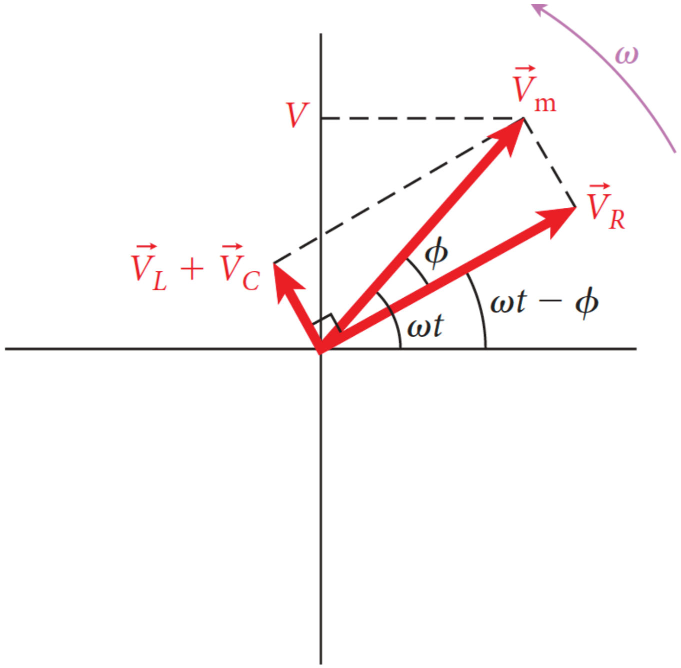
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

- The current is given by

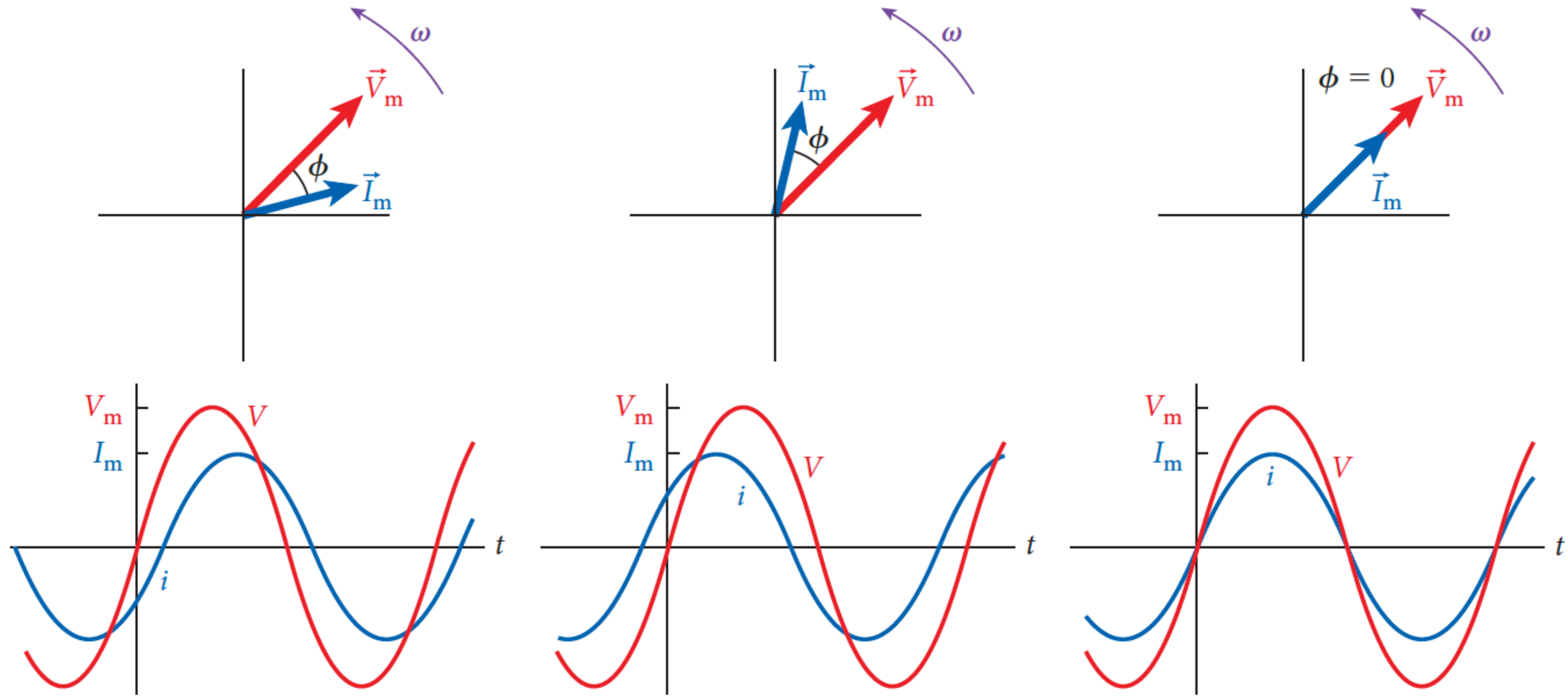
$$I_m = \frac{V_m}{Z} = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

- The phase is given by

$$\phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$



Driven RLC Circuit



$X_L = X_C$ $\Phi = 0$

$X_L > X_C$

Inductive

$X_C > X_L$

Capacitive

Resonance frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Power

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} \qquad V_{\text{rms}} = \frac{V_{\text{m}}}{\sqrt{2}}$$

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{V_{\text{rms}}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

- Average power dissipated:

$$\langle P \rangle = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}}{Z} I_{\text{rms}} R = I_{\text{rms}} V_{\text{rms}} \frac{R}{Z}$$

- Phase:

$$\cos\phi = \frac{V_R}{V_{\text{m}}} = \frac{IR}{IZ} = \frac{R}{Z}$$

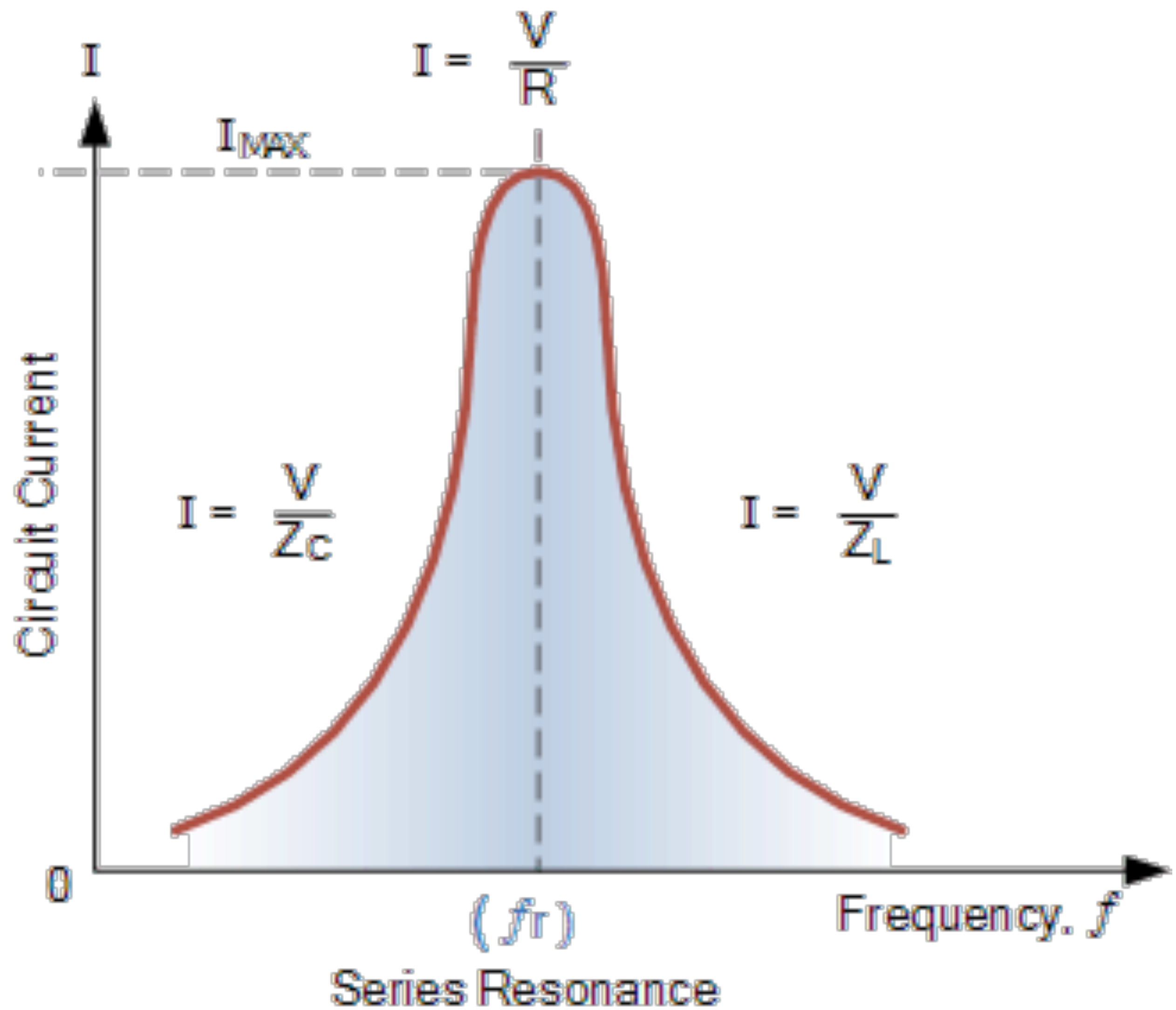
Power

- We can then express the power in the circuit as
  $\langle P \rangle = I_{\text{rms}} V_{\text{rms}} \cos\phi$
- We can see that the maximum power is dissipated when $\phi = 0$, i.e. at the resonance frequency

- We call $\cos\phi$ the power factor
- Quality factor of a series RLC circuit:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

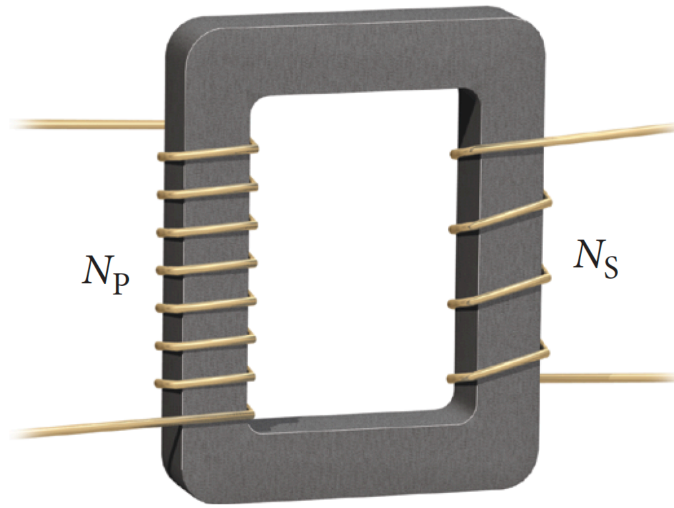
Series RLC circuit frequency dependence



Transformer

- Primary voltage V_P varies sinusoidally.
- Secondary voltage V_P is given by

$$\frac{V_P}{N_P} = \frac{V_S}{N_S} \quad \Rightarrow \quad V_S = V_P \frac{N_S}{N_P}$$



- Power and current:

$$P_P = I_P V_P = P_S = I_S V_S \qquad I_S = I_P \frac{V_P}{V_S} = I_P \frac{N_P}{N_S}$$

- Effective primary resistance for a load resistance R:

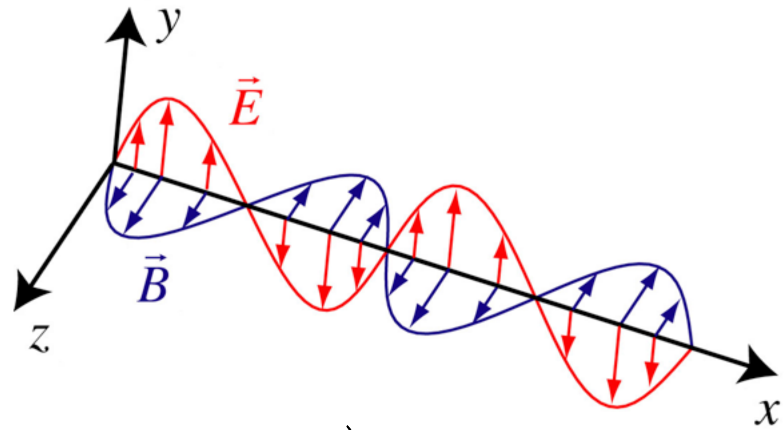
$$R_P = \frac{V_P}{I_P} = V_P \left(\frac{N_P}{N_S}\right)^2 \frac{R}{V_P} = \left(\frac{N_P}{N_S}\right)^2 R$$

Electromagnetic Waves

- Plane-wave ansatz to solve the Maxwell equations

$$E(\vec{r},t) = E_{max} \sin\left(kx - \omega t\right)$$

$$B(\vec{r},t) = B_{max} \sin\left(kx - \omega t\right)$$



$$\frac{\omega}{k} = \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} = f\lambda = c \quad \left(c \text{ is the speed of light}\right)$$

$$\frac{E_{max}}{B_{max}} = \frac{\omega}{k} = c \quad \Rightarrow \quad \frac{E}{B} = c$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Faraday

Ampere Maxwell

Energy Transport

- The rate of energy transported by an electromagnetic wave is usually defined as

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \qquad \text{Poynting vector} \qquad \mathrm{W/m^2}$$

- The instantaneous power per unit area

$$S = \left|\vec{S}\right| = \left(\frac{\text{power}}{\text{area}}\right)_{\text{instantaneous}}$$

- Instantaneous power per unit area of an electromagnetic wave

$$S = \frac{1}{\mu_0}EB \qquad S = \frac{1}{c\mu_0}E^2$$

- Intensity I of the wave given by

$$I = S_{ave} = \left(\frac{\text{power}}{\text{area}}\right)_{ave}$$

$$I = \frac{1}{c\mu_0}E_{rms}^2 \qquad \left(E_{rms} = E_{\max}/\sqrt{2}\right)$$

Energy Density and Radiation Pressure

- The energy density of an electric and a magnetic field is given by

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \qquad u_B = \frac{1}{2\mu_0}B^2$$

- The radiation pressure, p_r, due to a totally absorbed electromagnetic wave is

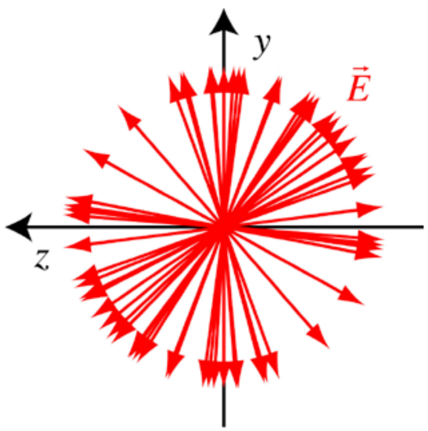
$$p_r = \frac{I}{c}$$

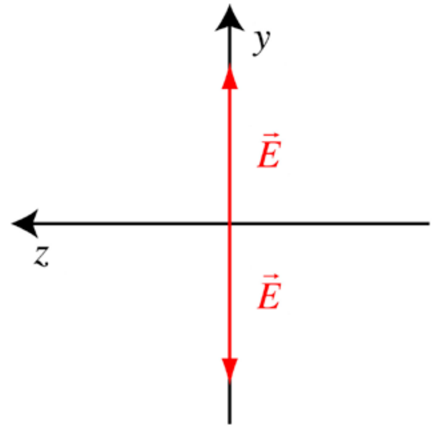
- The radiation pressure, p_r, due to a totally reflected electromagnetic wave is

$$p_r = \frac{2I}{c}$$

Polarization

- Unpolarized light polarized light





- We can change unpolarized light into polarized light by passing it through a polarizer. A polarizer is characterized by its polarizing direction
- The intensity *I* of the light - initially unpolarized - passing through the polarizer is given by

$$I = \frac{1}{2} I_0$$

- Polarized light passing through a polarizer:
- I_0 is the initial intensity and θ is the angle between the polarization of the incident light and the polarizing angle of the polarizer

$$I = I_0 \cos^2 \theta$$

Radiation Pressure

- Poynting vector S $S = \frac{1}{c\mu_0} E^2 = \frac{\text{Power}}{\text{area}}$
- Electromagnetic wave incident on an object
- Force is

$$F = \frac{\Delta p}{\Delta t} = \frac{IA}{c} \qquad \text{with} \qquad I = \frac{1}{c\mu_0} E_{\text{rms}}^2 \qquad \left(E_{\text{rms}} = E_{\max} / \sqrt{2}\right)$$

- Intensity A, speed of light c
- If wave is absorbed, pressure is

$$p_r = \frac{I}{c}$$

- If wave is reflected, pressure is

$$p_r = \frac{2I}{c}$$

Solar Stationary Satellite

- Suppose researchers want to place a satellite above the north pole of the Sun and stationary with respect to the Sun in order to study its long-term rotational characteristics
- The satellite will have a totally reflecting solar sail and be located at a distance of $1.50 \cdot 10^{11}$ m from the center of the Sun
- The intensity of sunlight at that distance is 1400 W/m^2
- The plane of the solar sail is perpendicular to a line connecting the satellite and the center of the Sun
- The mass of the satellite and sail is 100.0 kg

PROBLEM

- What is the required area of the solar sail?

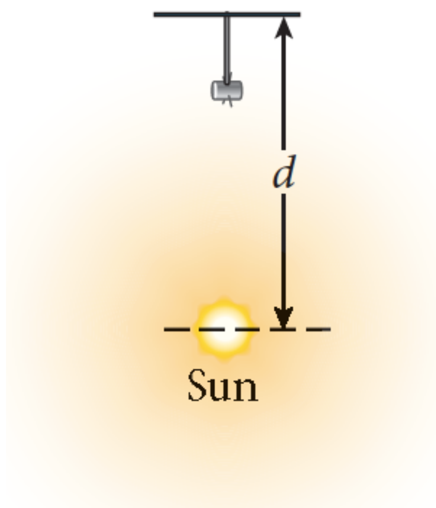
Solar Stationary Satellite

SOLUTION

THINK

- In an equilibrium position for the satellite, the area of the solar sail times the radiation pressure from the Sun produces a force that is balanced by the gravitational force between the satellite and the Sun
- We can equate these two forces and solve for the area of the solar sail

SKETCH



Solar Stationary Satellite

RESEARCH

- The satellite will be stationary if the force of gravity is balanced by the force from the radiation pressure

$$F_g = F_{rp}$$

- The force corresponding to the radiation pressure from sunlight is equal to the radiation pressure times the area of the solar sail

$$F_{rp} = p_r A$$

- The radiation pressure can be expressed in terms of the intensity of the sunlight incident on the totally reflecting solar sail

$$p_r = \frac{2I}{c}$$

Solar Stationary Satellite

- The force of gravity between the satellite and the Sun is

$$F_g = G\frac{mm_{\text{Sun}}}{d^2}$$

SIMPLIFY

- Combining our equations gives us

$$\left(\frac{2I}{c}\right)A = G\frac{mm_{\text{Sun}}}{d^2}$$

- Solving for the area of the solar gives us

$$A = G\frac{cmm_{\text{Sun}}}{2Id^2}$$

Solar Stationary Satellite

CALCULATE

- Putting in our numerical values we get

$$A = \left(6.67 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}\right) \frac{\left(3.00 \cdot 10^8 \text{ m/s}\right)\left(100.0 \text{ kg}\right)\left(1.99 \cdot 10^{30} \text{ kg}\right)}{2\left(1400 \text{ W/m}^2\right)\left(1.50 \cdot 10^{11} \text{ m}\right)^2}$$

$$A = 63{,}206.2 \text{ m}^2$$

ROUND

- We round our result to three significant figures

$$A = 6.32 \cdot 10^4 \text{ m}^2$$

DOUBLE-CHECK

- If the solar sail were circular, the radius of the sail would be

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{6.32 \cdot 10^4 \text{ m}^2}{\pi}} = 142 \text{ m}$$

Solar Stationary Satellite

- We can relate the thickness t of the sail times the density ρ of the material from which the sail is constructed to the mass per unit area of the sail

$$t\rho = \frac{m}{A}$$

- If the sail were composed of kapton (ρ = 1420 kg/m^3) and had a mass of 75 kg, the thickness of the sail would be

$$t = \frac{m}{\rho A} = \frac{75 \text{ kg}}{\left(1420 \text{ km/m}^3\right)\left(6.32 \cdot 10^4 \text{ m}^2\right)} = 0.836\ \mu\text{m}$$

- Kapton is developed to remain stable in the wide range of temperatures found in space; from near absolute zero to over 600 K
- Current production techniques cannot produce kapton this thin, but it may be possible in the near future

Polarizer

- Unpolarized light hits a polarizer: Intensity

$$I = \frac{1}{2} I_0$$

- Polarized light hits a polarizer: Intensity

$$I = I_0 \cos^2 \theta$$

- θ is the angle between the light polarization and the direction of the polarizer
- Law of Malus