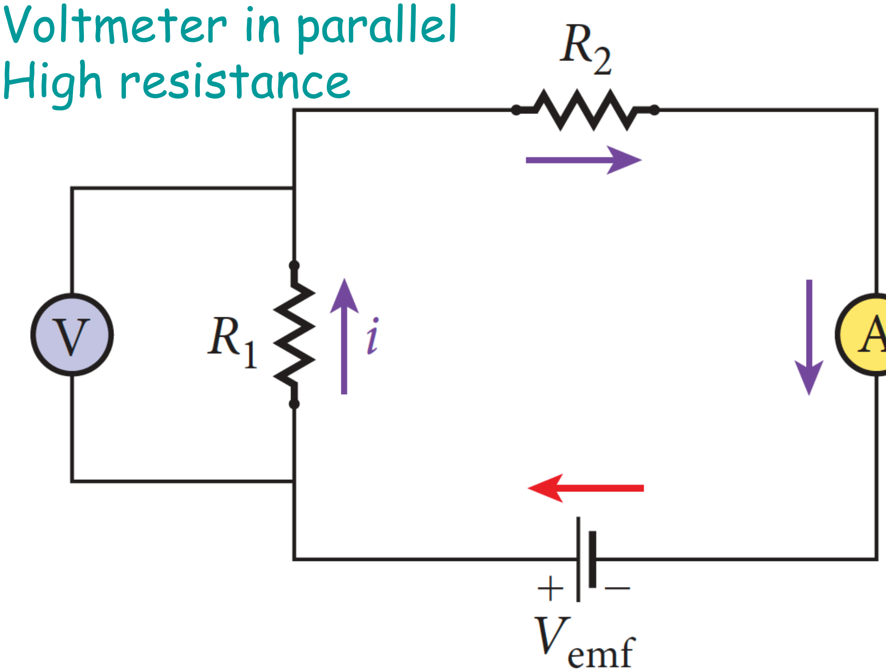
Direct Current Circuits

Ammeters and Voltmeters

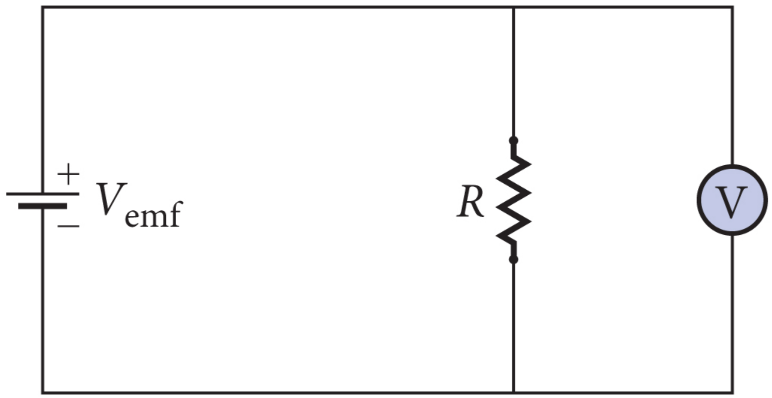
- A device used to measure current is called an **ammeter**
- A device used to measure potential difference is called a **voltmeter**
- To measure the current, the ammeter must be placed in the circuit in *series*
- To measure the potential difference, the voltmeter must be wired in *parallel* with the component across which the potential difference is to be measured

Voltmeter in parallel
High resistance



Ammeter in series
Low resistance

Real-life Voltmeter in a Simple Circuit



- Consider a simple circuit consisting of a source of emf with potential difference $V_{emf} = 150.$ V and a resistor with resistance $R = 100$ kΩ.
- A voltmeter with resistance $R_V = 10.0$ MΩ is connected across the resistor.

PROBLEM 1:

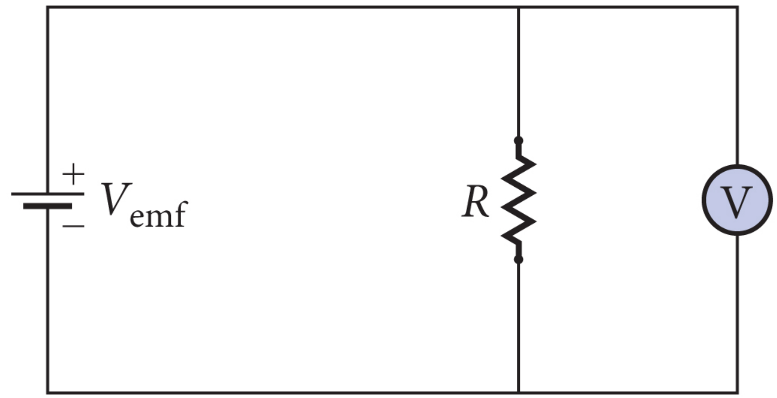
- What is the current in the circuit before the voltmeter is connected?

SOLUTION 1:

- Ohm's Law allows us to calculate the current before:

$$i = \frac{V_{emf}}{R} = \frac{150.\text{ V}}{100.\cdot 10^3\ \Omega} = 1.50 \cdot 10^{-3}\text{ A} = 1.50\text{ mA}$$

Real-life Voltmeter in a Simple Circuit



PROBLEM 2:

- What is the current in the circuit after the voltmeter is connected?

SOLUTION 2:

- The equivalent resistance of the resistor and the voltmeter connected in parallel is given by:

$$\frac{1}{R_{eq}} = \frac{1}{R} + \frac{1}{R_V}$$

- The equivalent resistance is:

$$R_{eq} = \frac{RR_V}{R + R_V} = \frac{(100.\cdot 10^3\ \Omega)(10.0\cdot 10^6\ \Omega)}{100.\cdot 10^3\ \Omega + 10.0\cdot 10^6\ \Omega} = 9.90\cdot 10^4\ \Omega = 99.0\ \text{k}\Omega$$

Real-life Voltmeter in a Simple Circuit

- The current after connecting the voltmeter is:

$$i = \frac{V_{\text{emf}}}{R} = \frac{150.\ \text{V}}{99.0 \cdot 10^{4}\ \Omega} = 1.52 \cdot 10^{-3}\ \text{A} = 1.52\ \text{mA}$$

- So the current in the circuit increases by 0.02 mA, when the voltmeter is connected because the parallel combination of the resistor and the voltmeter has a lower resistance than that of the resistor alone.
- The effect is small.
- The larger the resistance of the voltmeter, the better.

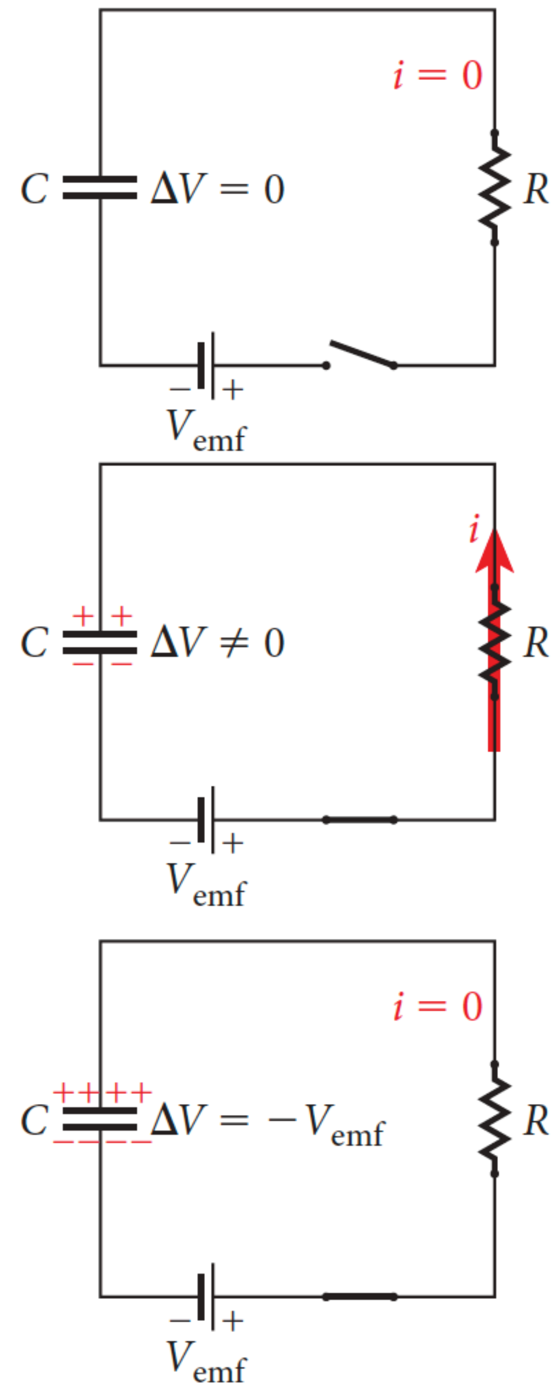
RC Circuits

- So far we have dealt with circuits containing sources of emf and resistors
- The currents in these circuits did not vary in time
- Now we will study circuits that contain capacitors as well as sources of emf and resistors
- These **RC circuits** have currents that vary with time
- Consider a circuit with
  - a source of emf, V_{emf},
  - a resistor R,
  - a capacitor C

Charging a Capacitor

- We start with the capacitor uncharged
- We close the switch and current begins to flow, building up opposite charges on the plates of the capacitor and creating a potential difference across the capacitor
- When the capacitor is fully charged, no more current flows
- The potential difference across the plates is equal to the potential difference provided by the source of emf
- The magnitude of the total charge on each plate is

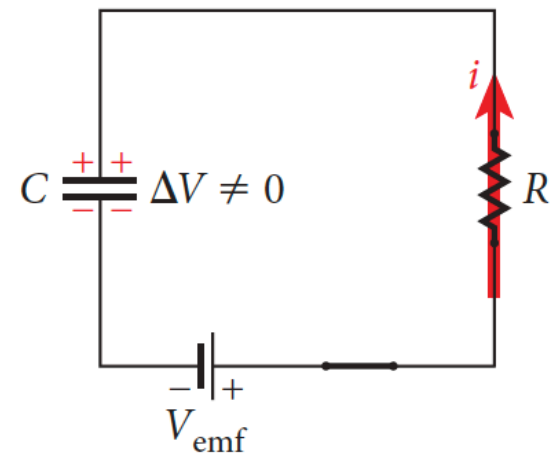
$$q_{tot} = CV_{emf}$$



Charging a Capacitor

- While the capacitor is charging, we can analyze the the current flowing by applying Kirchhoff's Loop Rule

$$V_{\text{emf}} - V_{\text{R}} - V_{\text{C}} = V_{\text{emf}} - i(t)R - \frac{q(t)}{C} = 0$$



- The change of the charge on the capacitor is the current

$$i(t) = \frac{dq(t)}{dt}$$

- We can rewrite our equation as

$$R\frac{dq(t)}{dt} + \frac{q(t)}{C} = V_{\text{emf}} \quad \text{or}$$

$$\frac{dq(t)}{dt} + \frac{q(t)}{RC} = \frac{V_{\text{emf}}}{R}$$ Differential equation relating q and $\frac{dq}{dt}$

The term V_c is negative since the top plate of the capacitor is connected to the positive - higher potential - terminal of the battery. Thus analyzing counter-clockwise leads to a drop in voltage across the capacitor!

Charging a Capacitor

- The solution for this differential equation is

$$q(t) = q_{\max}\left(1 - e^{-\frac{t}{\tau}}\right)$$

- The constant τ is the **time constant** given by
  $\tau = RC$
- The constant $q_{\max}$ is given by

$$q_{\max} = CV_{\text{emf}}$$

- So we can write the solution to our differential equation

$$q(t) = CV_{\text{emf}}\left(1 - e^{-\frac{t}{RC}}\right)$$

Math Reminder: $\frac{d}{dx}e^{ax} = a \cdot e^{ax}$

Charging a Capacitor

- At $t = 0$ we have

$$q(0) = CV_{\text{emf}}\left(1 - e^{-\frac{0}{RC}}\right) = 0$$

- At $t = \infty$ we have

$$q(\infty) = CV_{\text{emf}}\left(1 - e^{-\frac{\infty}{RC}}\right) = CV_{\text{emf}}$$

- The current flowing through the circuit is the time derivative of the charge

$$i(t) = \frac{dq(t)}{dt} = \left(\frac{V_{\text{emf}}}{R}\right)e^{-\frac{t}{RC}}$$

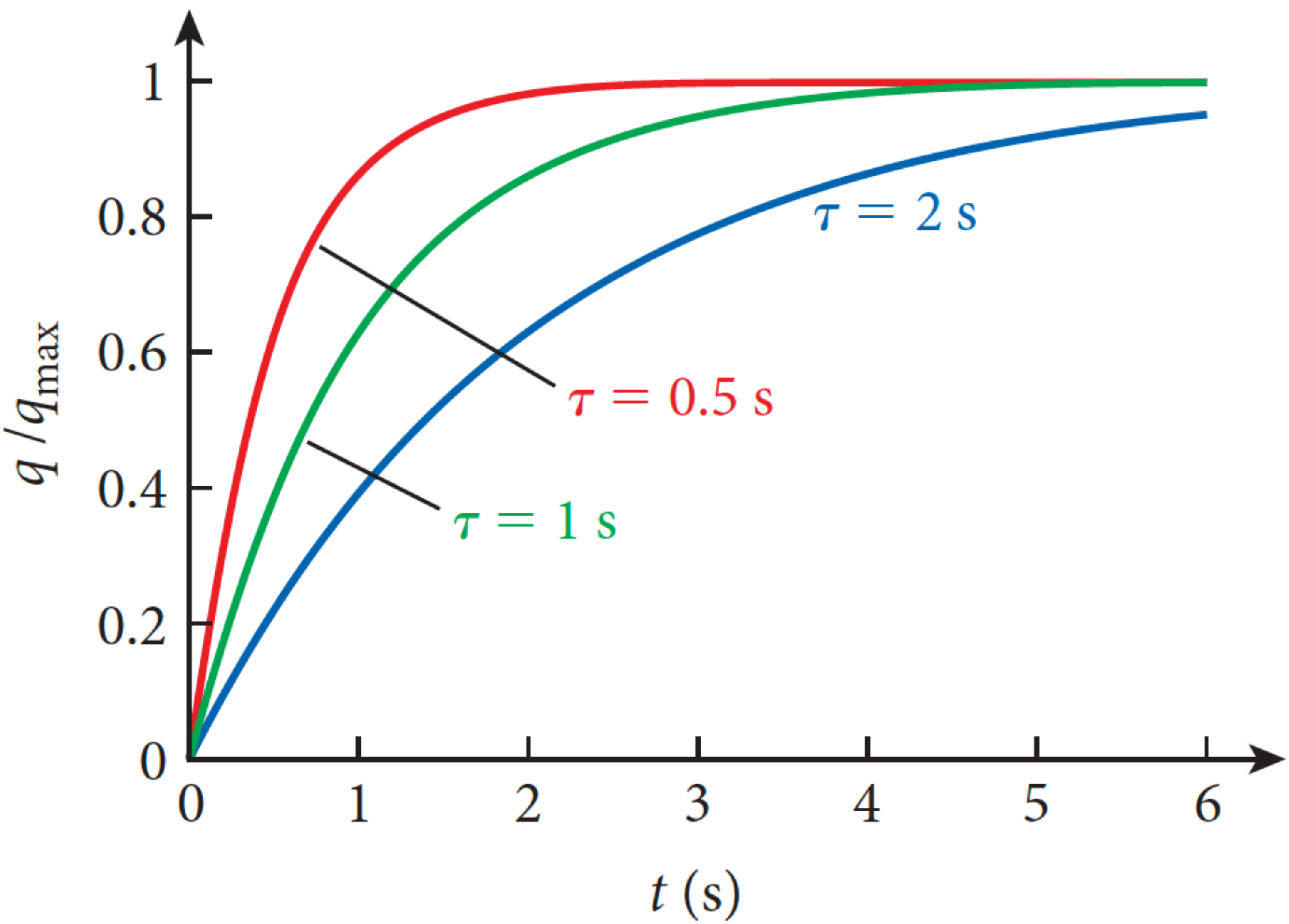
- At $t = 0$ we have

$$i(0) = \frac{V_{\text{emf}}}{R}$$

- At $t = \infty$ we have

$$i(\infty) = \left(\frac{V_{\text{emf}}}{R}\right)e^{-\frac{\infty}{RC}} = 0$$

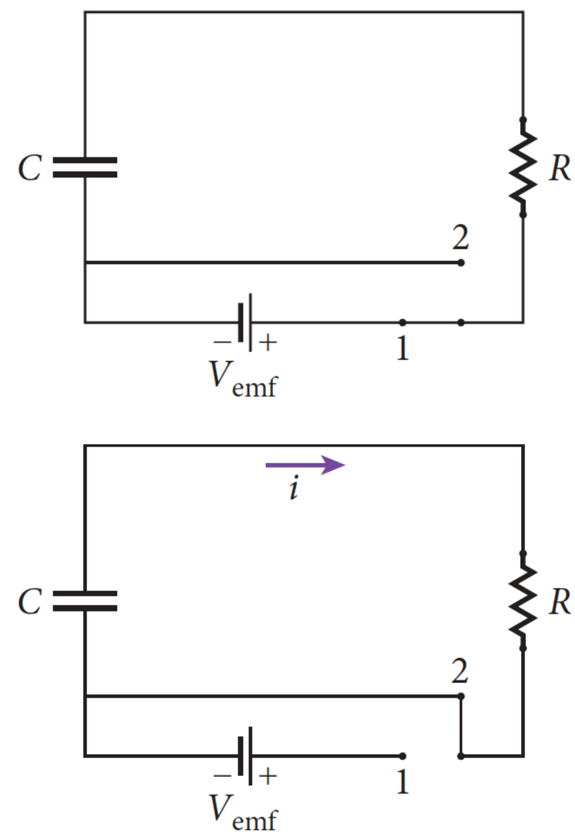
Charging a Capacitor



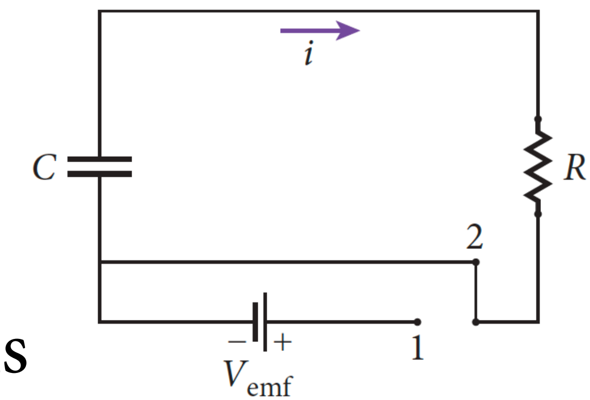
Discharging a Capacitor

- Now we consider a circuit containing only one resistor and a fully charged capacitor
- The charge on the capacitor is q_{max}
- Now we short the resistor across the capacitor by moving the switch from position 1 to position 2
- Current flows until the capacitor is completely discharged
- While the capacitor is discharging, Kirchhoff's Loop Rule gives us

$$-i(t)R - V_C = -i(t)R - \frac{q(t)}{C} = 0$$



Discharging a Capacitor



- Using the definition of current, we can rewrite our equation as

$$R\frac{dq(t)}{dt}+\frac{q(t)}{C}=0$$

- The solution to this differential equation is

$$q(t)=q_{max}e^{-\frac{t}{RC}}$$

- At $t = 0$ we have

$$q(0)=q_{max}e^{-\frac{0}{RC}}=q_{max}$$

- At $t = \infty$ we have

$$q(\infty)=q_{max}e^{-\frac{\infty}{RC}}=0$$

Discharging a Capacitor

- The current flowing through the circuit is the time derivative of the charge

$$i(t) = \frac{dq(t)}{dt} = -\frac{(q_{\max})}{RC} e^{-\frac{t}{RC}}$$

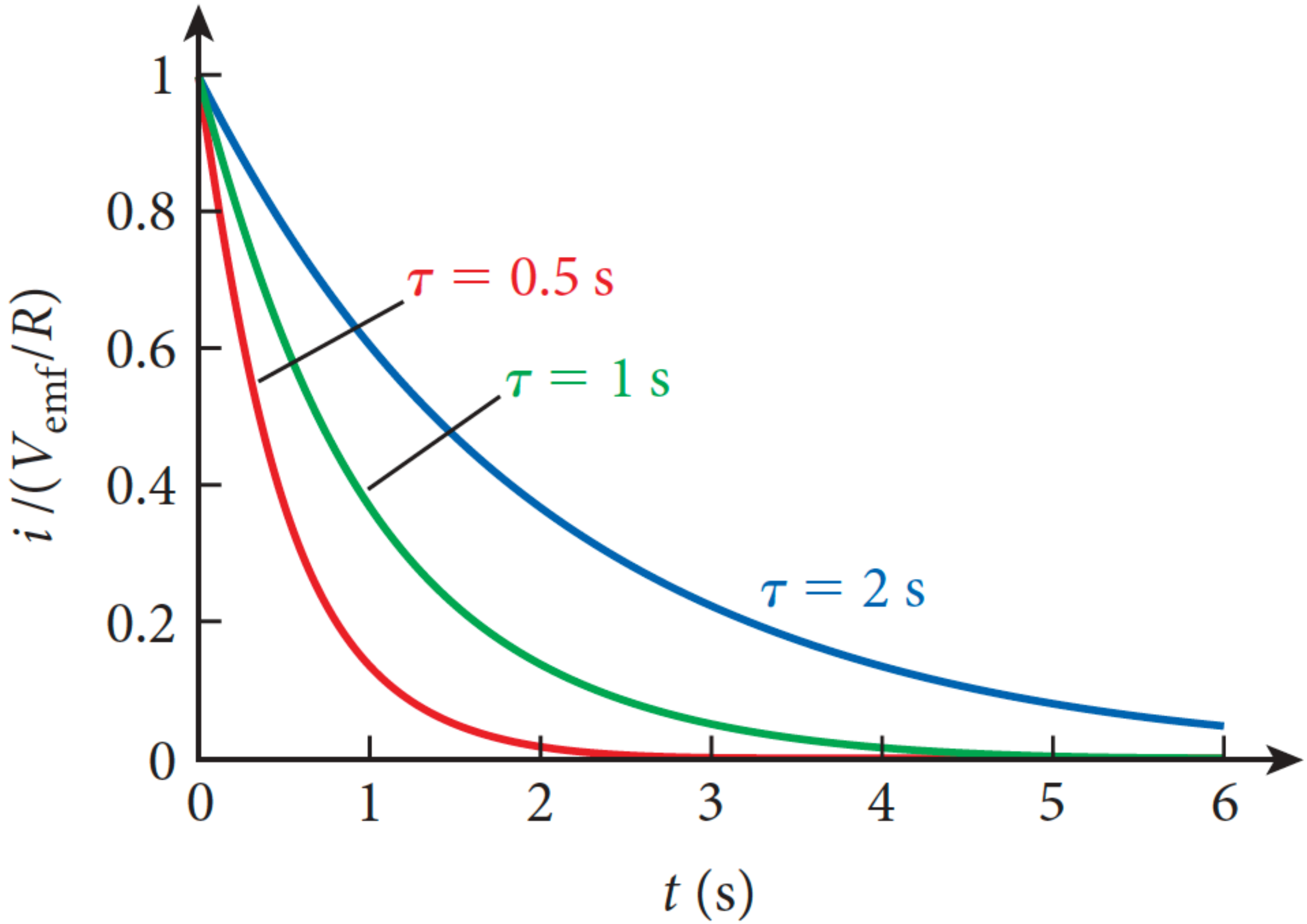
- At $t = 0$ we have

$$i(0) = -\frac{(q_{\max})}{RC} e^{-\frac{0}{RC}} = -\frac{(q_{\max})}{RC}$$

- At $t = \infty$ we have

$$i(t) = -\frac{(q_{\max})}{RC} e^{-\frac{\infty}{RC}} = 0$$

Discharging a Capacitor



Time Required to charge a Capacitor

- Consider a circuit consisting of a 12.0 V battery, a 50.0 Ω resistor, and a 100.0 μF capacitor wired in series
- The capacitor is initially completely discharged

- **PROBLEM**
- How long after the circuit is closed will it take to charge the capacitor to 90% of its maximum charge?

SOLUTION

- The charge on the capacitor as a function of time is given by

$$q(t) = q_{\max}\left(1 - e^{-\frac{t}{RC}}\right)$$

Time Required to charge a Capacitor

- We want to know the time until $q(t)/q_{max} = 0.90$

$$\left(1-e^{-\frac{t}{RC}}\right)=\frac{q(t)}{q_{max}}=0.90 \quad \Rightarrow \quad 0.10=e^{-\frac{t}{RC}}$$

- Taking the natural log of both sides gives us

$$\ln(0.10)=\ln\left(e^{-\frac{t}{RC}}\right) \quad \Rightarrow \quad \ln(0.10)=-\frac{t}{RC}$$

- So the time to discharge the capacitor to 90% of its maximum charge is

$$t=-RC\ln(0.10)=-(50.0\ \Omega)\left(100.0\cdot 10^{-6}\ \text{F}\right)(-2.30)$$

$$t=0.0115\ \text{s}=11.5\ \text{ms}$$

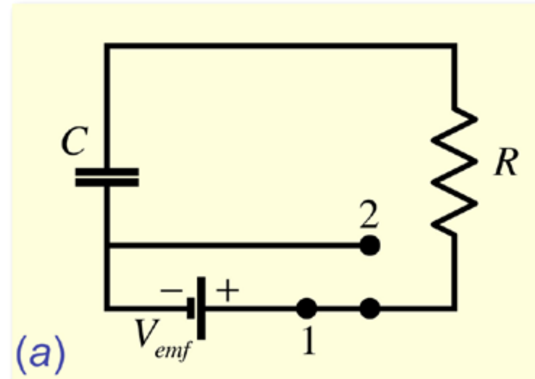
Math Reminder: $\ln(e^x)=x$

Summary: RC Circuit

- Charging a capacitor:

$$q(t) = q_0 \left(1 - e^{-\frac{t}{\tau}}\right)$$

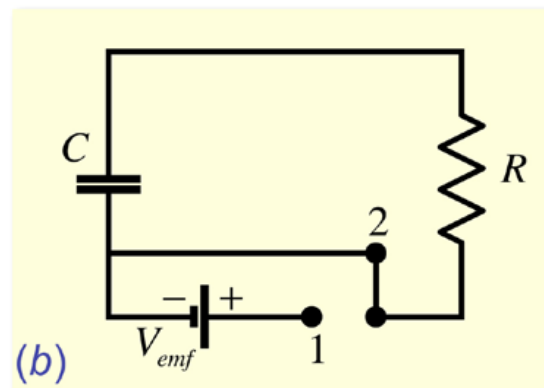
$$i = \frac{dq}{dt} = \left(\frac{V_{emf}}{R}\right) e^{\left(-\frac{t}{RC}\right)}$$



- Discharging a capacitor:

$$q = q_0 e^{\left(-\frac{t}{RC}\right)}$$

$$i = \frac{dq}{dt} = \left(\frac{q_0}{RC}\right) e^{\left(-\frac{t}{RC}\right)}$$



- Time constant: $\tau = RC$

Rate of Energy Storage in a Capacitor

- A resistor with $R = 2.50\ \text{M}\Omega$ and a capacitor with $C = 1.25\ \mu\text{F}$ are connected in series with a battery for which $V_{\text{emf}} = 12.0\ \text{V}$
- At $t = 2.50$ s after the circuit is closed, what is the rate at which energy is being stored in the capacitor?

SOLUTION

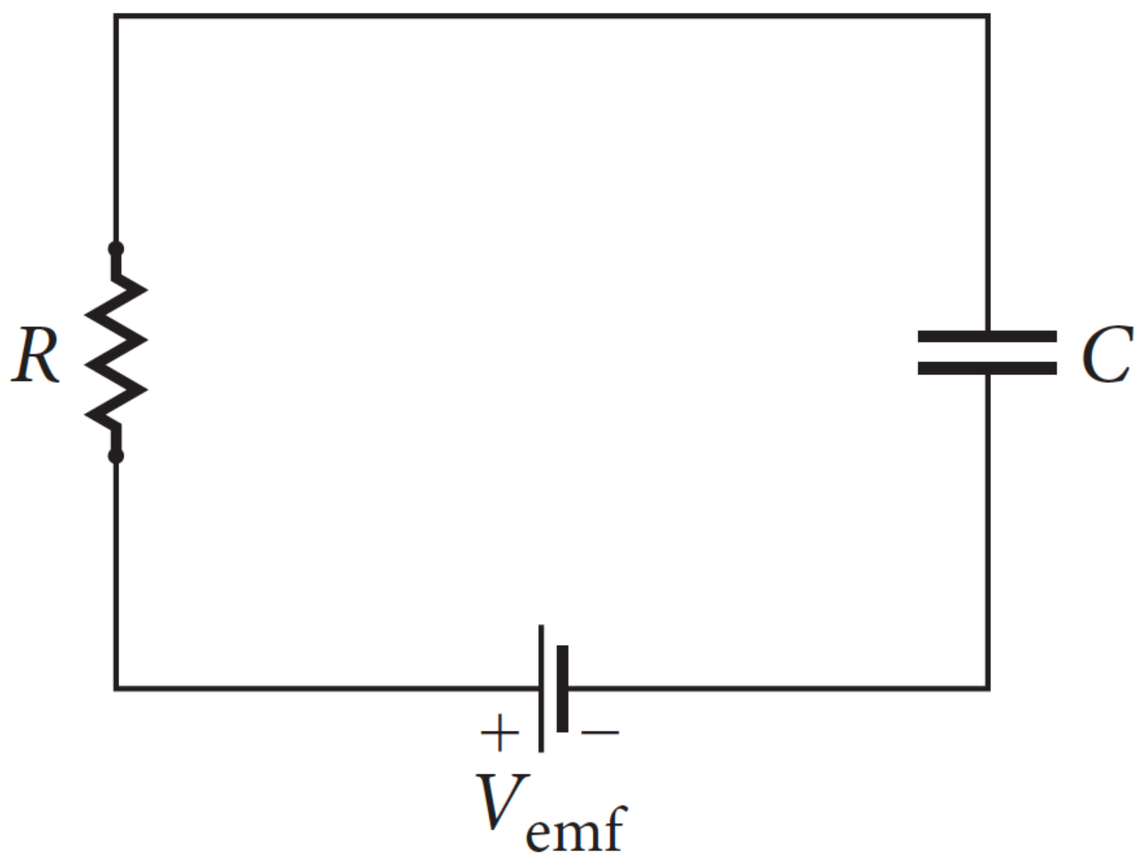
THINK

- When the circuit is closed, the capacitor begins to charge
- The rate at which energy is stored in the capacitor is given by the time derivative of the amount of energy stored in the capacitor, which is a function of the charge on the capacitor

Rate of Energy Storage in a Capacitor

SKETCH

- The circuit contains a battery, a resistor, and a capacitor in series.



Rate of Energy Storage in a Capacitor

RESEARCH

- The charge on the capacitor as a function of time is

$$q(t) = CV_{\text{emf}}\left(1 - e^{-\frac{t}{RC}}\right)$$

- The energy stored in a capacitor with charge q is

$$U = \frac{1}{2}\frac{q^2}{C}$$

- The time derivative of the stored energy is

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C}\right) = \frac{q(t)}{C}\frac{dq(t)}{dt}$$

- The time derivative of the charge is the current

$$i(t) = \frac{dq(t)}{dt} = \left(\frac{V_{\text{emf}}}{R}\right)e^{-\frac{t}{RC}}$$

Rate of Energy Storage in a Capacitor

SIMPLIFY

- Combining our equations

$$\frac{dU}{dt}=\frac{q(t)}{C}i(t)=\frac{CV_{\text{emf}}\left(1-e^{-\frac{t}{RC}}\right)}{C}\left(\frac{V_{\text{emf}}}{R}\right)e^{-\frac{t}{RC}}$$

$$\frac{dU}{dt}=\frac{V_{\text{emf}}^2}{R}e^{-\frac{t}{RC}}\left(1-e^{-\frac{t}{RC}}\right)$$

CALCULATE

- We first calculate the time constant

$$RC=\left(2.50\cdot10^{6}\ \Omega\right)\left(1.25\cdot10^{-6}\ \text{F}\right)=3.125\ \text{s}$$

Rate of Energy Storage in a Capacitor

- Now we calculate the rate of change of energy stored in the capacitor

$$\frac{dU}{dt}=\frac{(12.0\text{ V})^2}{2.50\cdot 10^6\ \Omega}e^{-\frac{2.50\text{ s}}{3.125\text{ s}}}\left(1-e^{-\frac{2.50\text{ s}}{3.125\text{ s}}}\right)=1.425\cdot 10^{-5}\text{ W}$$

DOUBLE-CHECK (Energy Conservation)

- The current at t = 2.50 is

$$i(t)=\left(\frac{12.0\text{ V}}{2.50\cdot 10^6\ \Omega}\right)e^{-\frac{2.50\text{ s}}{3.125\text{ s}}}=2.16\cdot 10^{-6}\text{ A}$$

Rate of Energy Storage in a Capacitor

- The rate of energy dissipation at $t = 2.50$ is

$$P = \frac{dU}{dt} = i^2 R = \left(2.16 \cdot 10^{-6}\ \text{A}\right)^2 \left(2.50 \cdot 10^{6}\ \Omega\right) = 1.16 \cdot 10^{-5}\ \text{W}$$

- The rate at which the battery delivers energy to the circuit at $t = 2.50$ s is

$$P = \frac{dU}{dt} = iV_{\text{emf}} = \left(2.16 \cdot 10^{-6}\,\text{A}\right)\left(12.0\ \text{V}\right) = 2.59 \cdot 10^{-5}\ \text{W}$$

- The energy supplied by the battery is equal to the sum of the energy dissipated in the resistor plus the energy stored in the capacitor