SOLAR CONVECTION: COMPARISON OF NUMERICAL SIMULATIONS AND MIXING-LENGTH THEORY

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ABSTRACT

We compare the results of realistic numerical simulations of convection in the superadiabatic layer near the solar surface with the predictions of mixing-length theory. We find that the peak values of such quantities as the temperature gradient, the temperature fluctuations, and the velocity fluctuations, as well as the entropy jump in the simulation, can be reproduced by mixing-length theory for a ratio of mixing length to pressure scale height \( x \approx 1.5 \). However, local mixing-length theory neither reproduces the profiles of these variables with depth nor allows penetration of convective motions into the overlying stable photosphere.

Subject headings: convection — methods: numerical — Sun: interior

1. INTRODUCTION

For over 40 yr, local mixing-length theory (MLT) has been used to calculate the convective transport of heat in models of stellar structure. MLT was invented to model convection in geophysical fluids (Taylor 1915). Decades later it was adapted to astrophysical settings by Vitense (1953) and Bohm-Vitense (1958). Because it is simple enough to use in evolution calculations, MLT has enjoyed widespread acceptance as a model for calculating the structure of stellar convection zones. Nevertheless, MLT suffers from a number of well-known limitations.

With a judicious choice of mixing length, MLT can give the correct entropy jump across the superadiabatic layer near the stellar surface, but it gives no information on dynamic phenomena, such as penetration and mixing into adjacent stable layers, or on the correlation of dynamic properties inside the convection zone. Hence it cannot adequately describe the overshooting from thermonuclear fusion–driven core convection zones and deep envelope convection zones in giant stars. Both of these effects are important in stellar evolution calculations. In addition, MLT requires specification of the mixing length, the average vertical distance that a convective fluid element travels before thermalizing with the ambient medium, but offers no guidance as to what this mixing length should be. The mixing length is commonly written as a multiple \( x \) of the local pressure scale height in the medium of interest, i.e., \( \ell = x H_p \), where the pressure scale height is \( H_p = P/\rho g \) (\( P \) is the pressure, \( \rho \) is the mass density, and \( g \) is the acceleration due to gravity).

Convection has also been modeled using three-dimensional numerical simulations by a number of authors (Nordlund 1985; Chan & Sofia 1989; Stein & Nordlund 1989; Cattaneo et al. 1991; Porter & Woodward 1994; Freytag, Ludwig, & Steffen 1996). Nordlund & Dravins (1990), Kim et al. (1996), and Freytag et al. (1996) have reported on detailed comparisons of the results of such simulations with the predictions of MLT. Numerical simulations and MLT produce very different pictures of convection. The simulations suggest that convection is driven on the scale of granulation by radiative cooling at the stellar surface. This process generates thin, cool, low-entropy downdrafts that descend through entropy neutral, smooth upflows and coalesce as they traverse the convection zone. The buoyancy work which drives the convection occurs primarily in these cool downdrafts. On the other hand, MLT represents convection as eddies of a single scale at any given depth and assumes that entropy fluctuations and driving occur equally in upflows and downflows.

In this paper, we compare the predictions of MLT for the values of several dynamic and thermodynamic quantities with the corresponding predictions of the three-dimensional numerical simulation of Stein & Nordlund. In particular, we examine the superadiabatic temperature gradient, the mean temperature gradient, the temperature, the entropy, the temperature fluctuations, and the velocity fluctuations. We determine the value of \( x \) that best brings the predictions of MLT into agreement with those of the simulation. However, we conclude that MLT fails to describe certain features of the depth dependence of the aforementioned physical quantities and dynamics that are clearly brought out in the simulation. Results from the Global Oscillation Network Group will provide critical tests of the reliability of both MLT and numerical simulations of stellar convection.

2. NUMERICAL SIMULATION

We model in detail the properties of convection from the solar surface to a depth of 2.5 Mm. The simulation solves the equations of conservation of mass, momentum, and internal energy. The upper part of the convective zone is highly stratified, and exhibits large fluctuations in pressure and density. For this reason, the code determines the logarithm of the density, instead of the density itself. This approach has the distinct advantage of reducing the magnitudes of the derivatives of the variables and is well suited for examining the properties of such a highly stratified medium. The pressure is obtained in a physically realistic manner from an equation of state that includes ionization and excitation of abundant atoms and formation of \( \mathrm{H}_2 \) molecules. Radiative heating is included in the simulation, since radiative energy transport plays a crucial role in controlling the structure of the upper convection zone. The numerical scheme is stabilized by introducing artificial diffusion.

Specifically, the following equations are solved:
convention of mass,
\[ \frac{d \ln \rho}{dt} = - u \cdot \nabla \ln \rho - \nabla \cdot u , \]
(1)

convention of momentum,
\[ \frac{du}{dt} = - u \cdot \nabla + g - \frac{P}{\rho} \nabla \ln P , \]
(2)

and conservation of internal energy per unit mass,
\[ \frac{de}{dt} = - u \cdot \nabla e - \frac{P}{\rho} \nabla \cdot u + Q_{\text{rad}} + Q_{\text{visc}} . \]
(3)

The pressure is calculated from a tabular equation of state (Gustafsson 1973). Radiative heating is given by
\[ Q_{\text{rad}} = \int \kappa I \, d\omega \, d\lambda . \]
(4)

Here \( \kappa = 1/(\text{mean free path}) \) is the opacity per unit volume of the medium. We assume LTE, so the source function is the Planck function, \( B \). The radiation intensity, \( I \), is determined by solving the transfer equation along rays through each of the grid locations on the horizontal plane at \( z = 0, \)
\[ \frac{dI}{d\tau} = I - B , \]
(5)

where \( \tau \) is the optical depth. We use one vertical and four slanted rays through each location and rotate the slanted rays at each time step. The viscous dissipation is calculated via
\[ Q_{\text{visc}} = \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_i}{\partial x_j} \right) . \]
(6)

Numerical diffusion of the form
\[ \frac{1}{\rho} \frac{\partial}{\partial x_j} \rho v \frac{\partial f}{\partial x_j} \]
(7)
is added to the fluid equations; here \( v \) is the diffusion coefficient. The diffusion coefficient is a sum of three terms: a term proportional to the fluid velocity to prevent ringing in advected quantities, a term proportional to the sound speed to stabilize low amplitude waves, and a term proportional to the velocity divergence to stabilize shocks.

Imposing computational boundaries where there are no real physical boundaries is a nontrivial matter. The horizontal boundaries are taken to be periodic, and the vertical boundaries are made as transmitting as possible without introducing instabilities. Since the convection is driven from the visible surface, which is within the computational domain, phenomena outside the computational realm should have only minimal effect upon the region within, making the imposed boundary conditions physically reasonable. For additional details, see Nordlund & Stein (1990).

3. MIXING LENGTH THEORY

Unlike the numerical simulation, MLT assumes a Boussinesq fluid. In MLT, convective elements travel a distance equal to one mixing length and then thermalize with the ambient medium. Further, MLT is local and hence requires that the mean velocity of convective elements go to zero at the boundaries of the unstable region. As a result, convective overshoot is ignored.

The convective flux is the excess energy in a convecting fluid parcel times its velocity. In MLT, the convective flux is given by (Cox & Giuli 1968):
\[ F_c = \frac{1}{2} \rho C_p \omega \left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right) . \]
(8)

Here \( C_p \) denotes the specific heat at constant pressure, \( T \) is the temperature of the fluid parcel, and \( \dot{T} \) is the average temperature of the ambient medium. The average velocity, \( \omega \), is calculated from the kinetic energy, which is the work done on the fluid parcel by the buoyancy force. The simulations have shown that in the superadiabatic region \( (d \ln \rho/d \ln T) \approx -0.02 \), so we have made the simplifying assumption of ignoring the effect of changes in the mean molecular weight. The resulting expression for \( \omega \) is then
\[ \omega = \frac{1}{2} \left( \frac{g}{T} \right)^{1/2} \left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right)^{1/2} . \]
(9)

The ratio of the excess energy of an element to the energy radiated away during its lifetime can be obtained from
\[ \frac{\left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right)}{\left( \frac{dT}{dz} \right)_{\text{AD}} - \frac{\dot{T}}{dz}} = \frac{3 \rho C_p \kappa \omega}{32 \sigma T^3} . \]
(10)

Here, \( dT/dz \) is the adiabatic temperature gradient, \( \kappa = 1/(\text{mean free path}) \) is the opacity per unit volume (not the mass) of the medium, and \( \sigma \) is the Stefan-Boltzmann constant.

In MLT the total flux is taken to be the sum of the radiative flux (in the diffusion approximation) and the convective flux (eqs. [18] and [8])
\[ F_{\text{tot}} = F_c + F_r . \]
(11)

MLT ignores the contribution of the kinetic energy flux \( (\approx 10\% \text{ of the total flux in our simulations}) \) and the viscous energy flux (which is indeed small in our simulations). We require that the MLT total flux at the upper boundary be equal to that of the simulation. Condition (11), along with equations (8), (9), and (10), can be used to derive the following cubic equation for the temperature gradient:
\[ \frac{C}{K} \left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right)^{3/2} + \left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right) + \frac{K}{C} \times \left( \frac{dT}{dz} - \frac{\dot{T}}{dz} \right)^{1/2} - \frac{F_{\text{TOT}}}{K} = 0 , \]
(12)

where
\[ C = \frac{1}{4} \rho C_p \left( \frac{g}{T} \right)^{1/2} l^2 \]
(13)

and
\[ K = (16 \sigma T^3)/(3 \kappa) . \]
(14)

The simulation and MLT both use the same equation of state and the same opacity tables. Our application of MLT is not entirely self-consistent, as the pressure and density stratification are taken from the simulation, which is, on average, close to hydrostatic equilibrium with the inclusion of the turbulent pressure from the convective motions \( (\approx 15\% \text{ of the gas pressure at its maximum}) \). This limitation
could be overcome by recalculating thermodynamic quantities using the temperature obtained from the MLT calculation, which is slightly different than in the simulation.

4. RESULTS

One can determine the following quantities from the solution to the MLT cubic equation:

convective flux,
\[ F_c = C \left( \frac{dT}{dz} - \frac{dT}{dz} \right)^{3/2}; \] (15)

superadiabatic gradient \( \beta = (dT/dz - d\mathcal{T}/dz)|_{AD}, \)
\[ \beta = \left( \frac{dT}{dz} - \frac{dT}{dz} \right) + \frac{K}{C} \left( \frac{dT}{dz} - \frac{dT}{dz} \right)^{1/2}; \] (16)

mean temperature gradient,
\[ \frac{dT}{dz} = \beta + \frac{dT}{dz}|_{AD}; \] (17)

radiative flux,
\[ F_r = -K \frac{dT}{dz}. \] (18)

We compare the MLT profiles with the corresponding horizontally and temporally averaged quantities obtained from the detailed numerical simulation. The MLT mean temperature profile is obtained by integrating the MLT mean temperature gradient over the height. The integration constant is chosen such that the temperature at the lower boundary matches that of the simulation. The entropy is calculated by integrating
\[ TdS = dE + PdV \] (19)
in both the simulation and MLT. Entropy is only defined up to an arbitrary additive constant; this free parameter is used to match the photospheric entropy minimum of the simulation and the MLT model.

We find that MLT exhibits a more rapid transition between the convective and radiative zones (Fig. 1). Increasing \( \alpha \) decreases the MLT superadiabatic gradient (Fig. 2), which has a broader profile than in the simulation. The upper boundary of the convective zone (the point where the superadiabatic gradient is zero) is located lower in MLT than in the simulation (Fig. 2) and is nearly independent of the value of \( \alpha \). It is also known that the lower boundary of the convective zone is located deeper for larger \( \alpha \).

The mean temperature profile, the peak temperature gradient, and the peak of the temperature fluctuations are all in better agreement with the results of the simulation for an \( \alpha \) of 1.5 than an \( \alpha \) of 1.0 (Figs. 3–5). As \( \alpha \) is increased, the maximum temperature gradient decreases and the region of large temperature gradients narrows (Fig. 4).

The simulation includes both vertical and horizontal velocities, whereas MLT determines only the vertical velocity. The density weighted, horizontally averaged vertical velocity is zero, because of mass conservation. It is appropriate, then, to compare the vertical velocity fluctuations with the MLT velocity. The maximum of the velocity fluctuations is nearly independent of \( \alpha \) (Fig. 6). The large discrepancy between the slopes of the velocity fluctuations in MLT and those of the simulation can be attributed to the fact that MLT uses a single eddy approximation, while the
F.4. Temperature gradient in MLT and the simulation. As \( \alpha \) increases, the maximum temperature gradient decreases and the region of large temperature gradient narrows.

F.5. Temperature fluctuations in MLT and the simulation. In MLT, \( \delta T = \frac{dT}{dz} - \frac{dT}{dz} \), the temperature fluctuation is the horizontally averaged rms fluctuation. The layer of large temperature fluctuations in MLT becomes narrower as \( \alpha \) increases.

F.6. Velocity fluctuations in MLT and the simulation. In MLT, the velocity fluctuation is the velocity calculated from eq. (9). In the simulations, the velocity fluctuation is the horizontally averaged rms fluctuation of the density weighted velocity, since by mass conservation \( \langle \rho u_x \rangle = 0 \). MLT velocities are insensitive to the value of \( \alpha \). Also note that in the simulation, as in the Sun, but unlike MLT, fluid overshoots into the photosphere.

Figures 5 and 6 clearly show that in the simulation, as for the Sun, fluctuations in temperature and velocity occur outside the convection zone because of penetration. Such fluctuations are not predicted by local MLT, which does not allow convective overshoot.

Finally, the MLT entropy jump across the superadiabatic region (the difference in entropy between its minimum in the photosphere and the adiabatic interior of the convection zone) matches that of the simulation much better for an \( \alpha \) of 1.5 than an \( \alpha \) of 1.0 (Fig. 7).

5. CONCLUSIONS

5.1. Calibration of \( \alpha \)

In order to determine which value of the free parameter \( \alpha \) is most appropriate for the upper region of the convective zone, one may compare the simulation profiles for \( \beta, \delta V, \delta T \) and \( S \) with the corresponding MLT profiles for different values of \( \alpha \). In general, one cannot fit both the peaks and the profiles of the curves. We chose to fit the peaks since differences in profile stem from underlying physical assumptions made by MLT and the simulation.

The choice of \( \alpha = 1.5 \) yields very good fits for the entropy jump, the temperature, the temperature gradient, and the temperature fluctuations. The superadiabatic gradient is fit better with a lower value of \( \alpha \). The velocity fluctuations and fluxes depend only weakly on \( \alpha \). Simulations of A stars and white dwarfs by Freytag et al. (1996) show that in these cases also the peak values of these variables in the superadiabatic layer are best fit by a value of \( \alpha \approx 1.5 \).

For stellar structure calculations, we need to know the entropy jump at the surface, that is, the difference between constant entropy in the interior of the convective zone and the entropy minimum in the photosphere. The jump obtained in the numerical simulations is reproduced by MLT for \( \alpha \approx 1.5 \). The depth of the convection zone also depends on \( \alpha \), and changes in the size of the convective zone affect the resonant cavity beneath the solar photosphere which traps acoustic waves. Such changes shift the eigenfrequencies of the resonant modes. A value of \( \alpha \approx 1.5 \) agrees with that needed to match the depth of the convection zone determined by helioseismology (Christensen-Dalsgaard et
al. 1991). If one chooses to match the profiles instead of the peaks, a somewhat higher value of $a$ is obtained. Hence, it may be possible to calibrate $a$ for different stellar models by comparing numerical simulations with MLT.

5.2. Differences between the Numerical Simulation and MLT

Although MLT can match the peaks of the gradients of the thermodynamic variables, the fluctuations of the dynamic and thermodynamic variables, and the entropy jump obtained from the simulations with an appropriate choice of the mixing length, it cannot match their profiles. Differences between the predictions of the simulation and MLT that we observe are attributed primarily to the local nature of MLT. Local analytic theories are inherently flawed because they do not account for the extremely thin layer at the boundary of the convection zone near the solar surface—a layer an order of magnitude smaller than the horizontal or vertical extent of the granules—in which radiative cooling removes entropy from the ascending fluid. In this layer, the fluid properties depend on the fluid’s past history as well as its local state. A consequence of the local nature of MLT is that all dynamical quantities related to convective motions go to zero at the boundary of the instability region, according to the Schwarzschild criterion as $d \ln T / d \ln P = (d \ln T / d \ln P)_{AD}$. A proper treatment of the convective boundary layer, including overshoot, requires a nonlocal formalism.

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