11. The Maxwell Equations
Chapter Summary

▶ Three of Maxwell's equations have been explored in detail in Chapters 1 – 10,

$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$	Gauss's Law for $\mathbf{E}(\mathbf{x}, t)$
$\nabla \cdot \mathbf{B} = 0$	Gauss's Law for $\mathbf{B}(\mathbf{x}, t)$
$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$	Faraday's Law

The fourth equation is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t; \qquad \text{Ampère-Maxwell Law}$$

the term $\epsilon_0 \partial \mathbf{E}/\partial t$ is the displacement current density.

▶ For systems with polarizable materials, it is convenient to introduce the fields **D** and **H**. Maxwell's equations are then

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad \nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \partial \mathbf{D}/\partial t$$

▶ We may simplify the solution of the equations formally by introducing potentials $\mathbf{A}(\mathbf{x}, t)$ and $V(\mathbf{x}, t)$, in terms of which

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\nabla V - \partial \mathbf{A}/\partial t.$$

▶ The field energy density and energy flux are

$$u = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \quad \text{and} \quad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B},$$

or, with linear polarizable materials,

$$u = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{B} \cdot \mathbf{H} \quad \text{and} \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

▶ Electromagnetic waves in vacuum

The Maxwell equations with $\rho = 0$ and $\mathbf{J} = 0$ have wave solutions. A harmonic plane wave has the form

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}. \end{aligned}$$

The field equations require that $\mathbf{E}_0$, $\mathbf{B}_0$, and $\mathbf{k}$ form an orthogonal triad of vectors with $\mathbf{E}_0 \times \mathbf{B}_0$ in the direction of $\mathbf{k}$. Also required are

$$\omega = ck \quad \text{and} \quad B_0 = \frac{E_0}{c},$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$. The constant c is the speed of light (and all other forms of electromagnetic radiation) in vacuum.