Electromagnetism
G. L. Pollack and D. R. Stump

2. Vector Calculus
Self-test answers

1. The calculation can be done in Cartesian coordinates (try it!) but it is simpler to use spherical polar coordinates. Note that $\mathbf{x} = r\widehat{\mathbf{r}}$. *(This is an important point—how to write* $\mathbf{x}$ *in spherical coordinates—so be sure to understand it. See Eq. (2.66) in Sec. 2.4.2.)* Therefore $C_r = r/(r^2 + a^2)^{3/2}$, and $C_\theta = C_\phi = 0$. Using Table 2.4 to calculate the divergence,

$$\boldsymbol{\nabla} \cdot \mathbf{C} = \frac{3a^2}{(r^2 + a^2)^{5/2}}.$$

The tangent curves are radial lines.

---

2. Let a be the radius of the circle C. Then

$$\oint_C \mathbf{F} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} F_\phi a d\phi = 2\pi a f(a).$$

Now calculate the integral of $\boldsymbol{\nabla} \times \mathbf{F}$ on the disk bounded by C. The curl, which may be calculated using either Cartesian or cylindrical coordinates, is

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r}\frac{d}{dr}\left[r f(r)\right] \widehat{\mathbf{k}}.$$

Thus

$$\begin{aligned} \int (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} &= \int_0^a \int_0^{2\pi} \frac{1}{r}\frac{d}{dr}\left[r f(r)\right] r dr d\phi \\ &= r f(r)|_0^a \times 2\pi = 2\pi a f(a). \end{aligned}$$

The loop integral of $\mathbf{F}$ is equal to the surface integral of $\boldsymbol{\nabla} \times \mathbf{F}$, as must be true by Stokes's theorem.

---

3. Both the flux integral and the integral of $\boldsymbol{\nabla} \cdot \mathbf{G}$ are equal to 4π.

---

4. The curl of $\mathbf{F}(\mathbf{x})$ may be calculated by several methods. Perhaps the simplest is to use vector identities. (See Sec. 2.2.4.) Start by simplifying the double cross product,

$$\mathbf{F}(\mathbf{x}) = \mathbf{c}r^2 - \mathbf{x}\,(\mathbf{c} \cdot \mathbf{x})$$

by Eq. (2.25). Now use the identity for $\boldsymbol{\nabla} \times (g\mathbf{F})$ in Table 2.2. The curl of the first term is

$$\begin{aligned} \boldsymbol{\nabla} \times \left(\mathbf{c}r^2\right) &= (\boldsymbol{\nabla} \times \mathbf{c})\, r^2 - \mathbf{c} \times \boldsymbol{\nabla} r^2 \\ &= -\mathbf{c} \times 2\mathbf{x}. \end{aligned}$$

(Note that $\boldsymbol{\nabla}\times\mathbf{c} = 0$ because $\mathbf{c}$ is constant, and $\boldsymbol{\nabla}(x^2+y^2+z^2) = 2\mathbf{x}$.) The curl of the second term is

$$\begin{aligned} \boldsymbol{\nabla}\times(\mathbf{x}\,\mathbf{c}\cdot\mathbf{x}) &= (\boldsymbol{\nabla}\times\mathbf{x})\;\mathbf{c}\cdot\mathbf{x} - \mathbf{x}\times\nabla\,(\mathbf{c}\cdot\mathbf{x}) \\ &= -\mathbf{x}\times\mathbf{c}. \end{aligned}$$

(Note that $\boldsymbol{\nabla}\times\mathbf{x} = 0$ and $\boldsymbol{\nabla}(\mathbf{c}\cdot\mathbf{x}) = \mathbf{c}$.) Then

$$\boldsymbol{\nabla}\times\mathbf{F} = -2\mathbf{c}\times\mathbf{x} + \mathbf{x}\times\mathbf{c} = -3\mathbf{c}\times\mathbf{x}.$$

---

5. The figure was made using Mathematica's `PlotVectorField`.