The Quadrupole Tensor, or Dyadic

The asymptotic electrostatic potential, far from a finite source $\rho(\mathbf{x})$, is given in Eq. (3.91) (in Sec. 3.8) as

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{r} + \frac{\widehat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \frac{\widehat{\mathbf{r}}\cdot\mathcal{Q}_2\cdot\widehat{\mathbf{r}}}{r^3} + \cdots\right\}, \tag{1}$$

where

$$q = \int \rho(\mathbf{x}')d^3x' = \text{ monopole scalar (charge)} \tag{2}$$

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x' = \text{ dipole vector (moment)} \tag{3}$$

$$\mathcal{Q}_2 = \int \tfrac{1}{2}\left(3\mathbf{x}'\mathbf{x}' - r'^2\mathbf{1}\right)\rho(\mathbf{x}')d^3x' = \text{ quadrupole tensor} \tag{4}$$

Scalars and vectors are familiar. The quadrupole factor $\mathcal{Q}_2$ is a tensor. In this note we'll study the quadrupole potential, which may be rewritten as

$$V^{(2)}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{x}\cdot\mathcal{Q}_2\cdot\mathbf{x}}{r^5}. \tag{5}$$

$\mathcal{Q}_2$ is written in (4) using dyadic notation. What is the meaning of $\mathbf{x}'\mathbf{x}'$? This is called a *dyadic product* (or *dyadic*, for short). And what is $\mathbf{1}$? This is the unit dyadic. To understand these terms, let's consider some general definitions.

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the dyadic product is a tensor

$$\mathcal{T} = \mathbf{AB}. \tag{6}$$

There is a notation here ($\mathbf{AB}$) that perhaps has not been encountered before. It will be defined presently in terms of Cartesian components.

▶ A vector $\mathbf{A}$ has 3 Cartesian components A_x, A_y, A_z, and may be represented as a column vector,

$$\mathbf{A} \;:\; \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}. \tag{7}$$

In suffix notation, $\mathbf{A}$ is denoted simply as A_i.

▶ A tensor $\mathcal{T}$ has 9 components, and may be represented as a 3×3 matrix,

$$\mathcal{T} \;:\; \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}. \tag{8}$$

In suffix notation, $\mathcal{T}$ is denoted simply as T_{ij}.

▶ The 9 elements of the dyad **AB** are products of a component of **A** and a component of **B**,

$$\mathbf{AB} \;:\; \begin{pmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{pmatrix}. \tag{9}$$

In suffix notation, **AB** is A_iB_j. We may regard **AB** as the matrix product of the column vector of **A** and the row vector of **B**

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \begin{pmatrix} B_x & B_y & B_z \end{pmatrix} = \begin{pmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{pmatrix}. \tag{10}$$

(In contrast, the product of row **A** and column **B** is the scalar product $\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$.)

▶ The matrix representation of the unit dyadic **1** is

$$\mathbf{1} \;:\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

In suffix notation **1** is δ_{ij}.

With the above definitions we are ready to express the quadrupole term in $V(\mathbf{x})$ in terms of Cartesian components. The meaning of $\mathbf{x}\cdot\mathcal{Q}_2\cdot\mathbf{x}$ is

$$\begin{aligned} \mathbf{x}\cdot\mathcal{Q}_2\cdot\mathbf{x} &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \mathcal{Q}_{xx} & \mathcal{Q}_{xy} & \mathcal{Q}_{xz} \\ \mathcal{Q}_{yx} & \mathcal{Q}_{yy} & \mathcal{Q}_{yz} \\ \mathcal{Q}_{zx} & \mathcal{Q}_{zy} & \mathcal{Q}_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= x^2\mathcal{Q}_{xx} + y^2\mathcal{Q}_{yy} + z^2\mathcal{Q}_{zz} \qquad (12) \\ &\quad + xy(\mathcal{Q}_{xy} + \mathcal{Q}_{yx}) + yz(\mathcal{Q}_{yz} + \mathcal{Q}_{zy}) + zx(\mathcal{Q}_{zx} + \mathcal{Q}_{xz}). \end{aligned}$$

It is simpler in suffix notation,

$$\mathbf{x}\cdot\mathcal{Q}_2\cdot\mathbf{x} = \sum_{i,j=1}^{3} x_i\mathcal{Q}_{ij}x_j. \tag{13}$$

Remember, $\mathcal{Q}_2$ depends on the charge distribution but it does not depend on **x** (the field point). Therefore $\mathbf{x}\cdot\mathcal{Q}_2\cdot\mathbf{x}$ has only terms that are quadratic or bilinear in x, y, z. The quadrupole contribution to $V(\mathbf{x})$ is order R^{-3} if x, y, z are of order R. (The monopole term is order R^{-1}, and the dipole term is order R^{-2}.)

The integral in (4) may also be written in terms of Cartesian coordinates. (In (4) drop the prime on $\mathbf{x}'$ for simplicity, but then don't confuse the source point (now **x**) with the field point!) Remember that $\mathcal{Q}_2$ is a tensor, so it has 9 components. The ij component is

$$\mathcal{Q}_{ij} = \int \tfrac{1}{2}\left(3x_ix_j - r^2\delta_{ij}\right)\rho(\mathbf{x})d^3x; \tag{14}$$

that is,

$$\begin{aligned} \mathcal{Q}_{xx} &= \tfrac{1}{2}\textstyle\int \left(3x^2 - r^2\right)\rho(\mathbf{x})d^3x, \\ \mathcal{Q}_{xy} &= \tfrac{1}{2}\textstyle\int \left(3xy\right)\rho(\mathbf{x})d^3x, \end{aligned}$$

and so forth. The Cartesian components of $\mathcal{Q}_2$ are these *second moments* of $\rho(\mathbf{x})$.

The inertia tensor in mechanics

A similar mathematics appears in rotational dynamics.[1] The kinetic energy of a rotating body is

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega} \tag{15}$$

where $\boldsymbol{\omega}$ = angular velocity, and $\mathcal{I}$ = inertia tensor. Written in Cartesian components,

$$\begin{aligned} T &= \tfrac{1}{2}\left[\omega_x^2 \mathcal{I}_{xx} + \omega_y^2 \mathcal{I}_{yy} + \omega_z^2 \mathcal{I}_{zz} \right. \\ &\quad \left. + \omega_x\omega_y(\mathcal{I}_{xy} + \mathcal{I}_{yx}) + \omega_y\omega_z(\mathcal{I}_{yz} + \mathcal{I}_{zy}) + \omega_z\omega_x(\mathcal{I}_{zx} + \mathcal{I}_{xz})\right]. \end{aligned} \tag{16}$$

The components of the inertia tensor (or dyadic) $\mathcal{I}$ are second moments of the mass density $\rho_m(\mathbf{x})$,

$$\begin{aligned} \mathcal{I}_{ij} &= \int \left(-x_i x_j + r^2\delta_{ij}\right)\rho_m(\mathbf{x}) d^3x; \\ \mathcal{I}_{xx} &= \int \left(-x^2 + r^2\right)\rho_m(\mathbf{x}) d^3x, \\ \mathcal{I}_{xy} &= \int \left(-xy\right)\rho_m(\mathbf{x}) d^3x, \end{aligned} \tag{17}$$

and so forth. Please note the similarities to Eqs. (12) and (14).

Interesting symmetric cases

- If $\rho(\mathbf{x})$ is spherically symmetric, i.e., $\rho(\mathbf{x}) = \rho(r)$, then $\mathcal{Q}_2 = 0$. (Can you prove this?)
- If $\rho(\mathbf{x})$ is axially symmetric, i.e., $\rho(\mathbf{x}) = \rho(r, z)$ (in cylindrical coordinates) then the matrix representation of $\mathcal{Q}_2$ has the form

$$\mathcal{Q}_2 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -2A \end{pmatrix}. \tag{18}$$

[Can you prove this, and derive a formula for A? Remember that in (14) the notation is $r^2 = x^2 + y^2 + z^2$, but in cylindrical coordinates $\rho(r, z)$ means $\rho(\sqrt{x^2 + y^2}, z)$.]

Final identity For any $\rho(\mathbf{x})$, the matrix of $\mathcal{Q}_2$ is traceless,

$$\mathcal{Q}_{xx} + \mathcal{Q}_{yy} + \mathcal{Q}_{zz} = 0. \tag{19}$$

(Can you prove this?)

Exercise Consider an axially symmetric pure quadrupole, as in (18), and assume $A > 0$. Show that the electric field directions are radially toward the origin for points on the positive or negative z axis, and radially away from the origin for points on the xy plane. Is that consistent with Figure 3.15?

[1] Leonardo da Vinci said "Nature is economical and her economy is quantitative." The same mathematics applies to different topics in physics.