The method of conjugate functions. (Section 5.4.) Let \( F(z) \) be an analytic function of the complex variable \( z = x + iy \). Either the real part or imaginary part of \( F(z) \) may be taken to be the potential function of a 2D electrostatic system, because these functions obey Laplace’s equation. A clever choice of \( F(z) \) will solve a 2D boundary value problem if \( \Re F \) or \( \Im F \) satisfies the boundary conditions of the system.

An interesting example is the function

\[
F(z) = \Phi_0 \arcsin \frac{z}{a}
\]

where \( \Phi_0 \) and \( a \) are real constants. Write \( F = \Phi_0 (u + iv) \). Then

\[
z = a \sin (u + iv) = a [\sin u \cos iv + \cos u \sin iv]
= a \sin u \cosh v + ia \cos u \sinh v.
\]

Because \( z = x + iy \) (where \( x \) and \( y \) are the Cartesian coordinates)

\[
x = a \sin u \cosh v, \quad (3)
y = a \cos u \sinh v. \quad (4)
\]

\( \Im F \) as potential

What boundary value problem is solved by the potential \( V(x, y) \equiv \Phi_0 v \)? Note that \( \sin^2 u + \cos^2 u = 1 \); therefore

\[
\frac{x^2}{[a \cosh v]^2} + \frac{y^2}{[a \sinh v]^2} = 1.
\]

In principle we could solve (5) for \( v(x, y) \).

For constant \( v \), Eq. (5) is the equation of an ellipse with semimajor axis \( a \cosh v \) along the \( x \) axis, and semiminor axis \( a \sinh v \) along the \( y \) axis. So, the equipotentials are confocal ellipses with foci at \( x = \pm a \). The equipotential \( V = 0 \) is the line segment

\[
y = 0 \quad \text{and} \quad -a \leq x \leq a.
\]

Hence \( V(x, y) = \Phi_0 v(x, y) \) is the potential function of a conducting strip, as illustrated in Fig.1. The equipotentials are the blue curves. From Section 5.4, the curves with constant \( u \) are the electric field lines. Points on these curves have

\[
\frac{x^2}{[a \sin u]^2} - \frac{y^2}{[a \cos u]^2} = 1
\]

(because \( \cosh^2 v - \sinh^2 v = 1 \)); the field lines are confocal hyperbolas, shown as the red curves in Fig.1.
Figure 1: Equipotential curves (blue) and field lines (red) of a charged conducting strip. The arrows correspond to the case of positive $\Phi_0$.

The electric field is $\mathbf{E} = -\nabla V$. The discontinuity of $\varepsilon_0 E_y$ across the strip gives the charge density $\sigma(x)$. Differentiating (5) with respect to $y$, with $x$ fixed, gives

$$0 = \frac{-2x^2}{a^2 \cosh^3 v} \sinh v dv + \frac{2y dy}{a^2 \sinh^2 v} - \frac{2y^2}{a^2 \sinh^3 v} \cosh v dv. \quad (8)$$

For points approaching the upper side of the strip ($y \to 0^+$) $\sinh v$ approaches 0; Eq.(4) implies

$$\frac{y}{\sinh v} = a \cos u,$$

or, by (3),

$$\frac{y}{\sinh v} = a \sqrt{1 - \sin^2 u} \sim \sqrt{a^2 - x^2} \text{ as } v \to 0.$$

Therefore, taking $y \to 0^+$ in (8), we find

$$0 \sim \frac{y dy}{\sinh v} - \frac{y^2}{\sinh^2 v} dv,$$

$$\left. \frac{dv}{dy} \right|_{y=0} = \lim_{y \to 0} \frac{\sinh v}{y} = \frac{1}{\sqrt{a^2 - x^2}},$$

$$E_y(x, 0^+) = \frac{-\Phi_0}{\sqrt{a^2 - x^2}}.$$

By symmetry, $E_y(x, 0^-)$ is $+\Phi_0/\sqrt{a^2 - x^2}$, so the surface charge density on the strip is

$$\sigma(x) = \frac{-2\varepsilon_0 \Phi_0}{\sqrt{a^2 - x^2}}. \quad (9)$$
Figure 2: Potential as a function of position on the $y$ axis.

The total charge per unit length (in the $z$ direction) is

$$\lambda = \int_{-a}^{a} \sigma(x)dx = -2\pi \epsilon_0 \Phi_0.$$  \hspace{1cm} (10)

This determines the constant $\Phi_0$ from the charge on the conducting strip.

The potential along the positive $y$ axis is

$$V(0, y) = \Phi_0 v(0, y) = \Phi_0 \text{arcsinh} \frac{y}{a}$$ \hspace{1cm} (11)

where we have set $u = 0$ in (3) and (4), so that $x = 0$. (The negative $y$ axis corresponds to $u = \pi$.) Note two interesting limits:

- For $y \ll a$, $V(0, y) \sim \Phi_0 y/a$, which corresponds to a constant field; in this limit the strip may be approximated by a charged plane.
- For $y \gg a$, $V(0, y) \sim \Phi_0 \ln(2y/a)$,\footnote{This follows from the interesting identity $\text{arcsinh} \xi = \ln \left( \xi + \sqrt{\xi^2 + 1} \right)$.} which corresponds to a charged wire; in this limit the strip may be approximated by a line.

Figure 2 shows $V(0, y)/\Phi_0$ as a function of $y/a$.

**RF as potential**

What boundary value problem is solved by the potential $V(x, y) \equiv \Phi_0 u$? The answer is illustrated in Figure 3: A conducting plane with a slot cut in it from $x = -a$ to $x = a$. The plates on opposite sides of the slot have opposite charges. The equipotential curves and electric field lines of the slot problem are the same as the electric field lines and equipotential curves, respectively, of the strip problem. The roles of $u$ and $v$ are reversed.
Figure 3: Equipotential curves (blue) and field lines (red) of a conducting plane with a slot. The two sides have opposite charges.