5. General Methods for Laplace’s Equation

Self-test answers

1. Using separation of variables and the boundary conditions, the potential must have the form

\[ V(x) = \sum_{m,n} c_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \cosh \gamma_{mn} z \cosh \gamma_{mn} L/2 \]

where \( \gamma_{mn}^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \). The end conditions require

\[ V_0 = \sum_{m,n} c_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) \]

which must be used to determine the constant coefficients \( c_{mn} \). The relevant orthogonality conditions involve integrals with \( 0 \leq x \leq a \) and \( 0 \leq y \leq b \). Then, projecting out \( c_{mn} \),

\[ c_{mn} = \frac{4}{ab} \int_0^a \int_0^b \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{a} \right) V_0 \, dx \, dy \]

\[ = \begin{cases} 
16V_0 & \text{if } m \text{ and } n \text{ are odd}, \\
\frac{\pi^2 ab mn}{1} & 0, \text{ otherwise.}
\end{cases} \]

2. For \( r \geq a \) the solution of Laplace’s equation is

\[ V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos \theta). \]

The term with \( \ell = 1 \) is

\[ A_1 r \cos \theta + B_1 \frac{\cos \theta}{r^2}. \]

We must set \( A_1 = -E_0 \) to get the potential \(-E_0z\) of the applied field. Then the boundary condition \( V(a, \theta) = 0 \) implies \( B_1 = E_0 a^3 \). No other \( \ell \) values are needed to satisfy the boundary conditions, so by the uniqueness theorem the potential is

\[ V(r, \theta) = -E_0 r \cos \theta + \frac{E_0 a^3}{r^2} \cos \theta. \]

3. (a) \( V(x, y) = x^2 - y^2 + \text{constant} \). The equipotentials are hyperbolas with asymptotes \( y = -x \) and \( y = x \). The boundary-value problem corresponding to this potential is charged conducting half planes on the orthogonal surfaces \( y = -x \) and \( y = x \).

(b) \( V(x, y) = 2xy + \text{constant} \). The equipotentials are hyperbolas with asymptotes \( y = 0 \) and \( x = 0 \). The boundary value problem corresponding to this potential is charged conducting half planes on the positive \( x \) and \( y \) axes.