5. General Methods for Laplace's Equation
Chapter Summary

• In a charge-free region of space, the electrostatic potential $V(\mathbf{x})$ satisfies $\boldsymbol{\nabla}^2 V = 0$. This PDE is Laplace's equation. Together with suitable boundary conditions it must determine $V(\mathbf{x})$.

• **Separation of variables.** For a system with rectangular boundaries, write $V(\mathbf{x}) = X(x)Y(y)Z(z)$. Depending on boundary conditions, $X(x)$ may be harmonic, $\sin kx$ or $\cos kx$; or it may be hyperbolic, $\sinh kx$ or $\cosh kx$. The same functions are possible for $Y(y)$ and $Z(z)$.

• The orthogonality relations used to determine Fourier coefficients, for the domain $-a \le x \le a$, are

$$\begin{aligned}
\int_{-a}^{a} \sin\frac{j\pi x}{a} \sin\frac{k\pi x}{a} dx &= a\delta_{jk} \\
\int_{-a}^{a} \cos\frac{j\pi x}{a} \cos\frac{k\pi x}{a} dx &= a\delta_{jk} \\
\int_{-a}^{a} \sin\frac{j\pi x}{a} \cos\frac{k\pi x}{a} dx &= 0
\end{aligned}$$

where j and k are integers.

• In spherical polar coordinates, the solutions to Laplace's equation by separation of variables, for a system with azimuthal symmetry, are

$$\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

where $\ell = 0, 1, 2, 3, \ldots$. The term $\propto r^\ell$ is finite at $r = 0$, and the term $\propto r^{-\ell-1}$ tends to 0 as $r \to \infty$. P_ℓ is the Legendre polynomial.

• For a 2-dimensional problem, $V = V(x, y)$, Laplace's equation may be solved by writing $V(x, y) = \operatorname{Re} F(z)$ where $z = x + iy$ and $F(z)$ is an analytic function. If $F(z)$ is constructed such that $\operatorname{Re} F(z)$ satisfies the boundary conditions then the uniqueness theorem guarantees that $\operatorname{Re} F(z)$ is the desired solution. The solution may also be constructed as $\operatorname{Im} F(z)$.