6. Electrostatics and Dielectrics
Self-test answers

1. (See Sec. 6.4.) Let A be the area of the plates, and d their separation. As usual, we'll neglect the edge effects. Then the electric field between the plates has the form $\mathbf{E} = E_0\widehat{\mathbf{k}}$, by symmetry, where $\widehat{\mathbf{k}}$ is the unit vector in the direction from $+Q$ to $-Q$.

The displacement field is $\mathbf{D} = D_0\widehat{\mathbf{k}}$. Apply Gauss's law to a pill box surrounding area A of one plate:

$$\oint \mathbf{D} \cdot d\mathbf{A} = Q_{\text{enclosed}} \qquad \Longrightarrow \qquad D_0 A = \sigma_{\text{free}} A.$$

(Q_{enclosed} is the free charge inside the pill box.) Thus

$$D_0 = \sigma_{\text{free}} = \frac{Q}{A}.$$

The constitutive equation is $\mathbf{D} = \epsilon\mathbf{E} = \kappa\epsilon_0\mathbf{E}$. Thus the electric field is

$$\mathbf{E} = \frac{\mathbf{D}}{\kappa\epsilon_0} = \frac{1}{\kappa}\frac{Q}{\epsilon_0 A}\widehat{\mathbf{k}}.$$

Note that the electric field is weaker by the factor κ^{-1}, because bound surface charge on the dielectric shields the free charge. The potential difference is

$$V = \int_{-}^{+} \mathbf{E} \cdot d\boldsymbol{\ell} = Ed = \frac{1}{\kappa}\frac{Qd}{\epsilon_0 A}.$$

The capacitance $C = Q/V$ is increased by the factor κ by the presence of the dielectric.

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2. (See Sec. 6.5.2.) Let a be the radius of the sphere. The electric field may be written as $-\boldsymbol{\nabla}V$ with

$$V(r,\theta) = \begin{cases} \dfrac{B\cos\theta}{r^2} - E_0 r\cos\theta & \text{for} \quad r > a \\ -Ar\cos\theta & \text{for} \quad r < a. \end{cases}$$

This potential function satisfies Laplace's equation. The boundary conditions must also be obeyed. The tangential component of $\mathbf{E}$ and the normal component of $\mathbf{D}$ are continuous at $r = a$:

$$-\frac{1}{a}\frac{\partial V_{\text{int}}}{\partial\theta} = -\frac{1}{a}\frac{\partial V_{\text{ext}}}{\partial\theta} \quad \text{which implies} \quad -A = \frac{B}{a^3} - E_0,$$
$$-\epsilon\frac{\partial V_{\text{int}}}{\partial r} = -\epsilon_0\frac{\partial V_{\text{ext}}}{\partial r} \quad \text{which implies} \quad \epsilon A = \epsilon_0\left(\frac{2B}{a^3} + E_0\right).$$

The solution is

$$\frac{B}{a^3} = \frac{\kappa - 1}{\kappa + 2} E_0 \quad \text{and} \quad A = \frac{3E_0}{\kappa + 2}.$$

The dipole potential for dipole moment $p\widehat{\mathbf{k}}$ is

$$V_{\text{dipole}} = \frac{p \cos\theta}{4\pi\epsilon_0 r^3}.$$

Comparing to the external potential, we see that the dipole moment is

$$p = 4\pi\epsilon_0 B = 4\pi\epsilon_0 a^3 \, \frac{\kappa - 1}{\kappa + 2} E_0.$$

The polarizability α is defined by $p = \alpha E_0$, so

$$\alpha = 4\pi\epsilon_0 a^3 \, \frac{\kappa - 1}{\kappa + 2}.$$

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3. (See Sec. 6.5.3.) There is an image charge solution, given in Eqs. (6.67)–(6.69). The field above the glass surface is

$$\mathbf{E}_{\text{above}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q\left(\mathbf{x} - h\widehat{\mathbf{k}}\right)}{|\mathbf{x} - h\widehat{\mathbf{k}}|^3} + \frac{q'\left(\mathbf{x} + h\widehat{\mathbf{k}}\right)}{|\mathbf{x} + h\widehat{\mathbf{k}}|^3} \right\}.$$

The field inside the glass is

$$\mathbf{E}_{\text{glass}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{q''\left(\mathbf{x} - h\widehat{\mathbf{k}}\right)}{|\mathbf{x} - h\widehat{\mathbf{k}}|^3}.$$

The image charges are

$$q' = -\frac{\kappa - 1}{\kappa + 1} q \quad \text{and} \quad q'' = \frac{2}{\kappa + 1} q.$$