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11. The Trigonometric Functions

11.1 REVIEW OF THE TRIGONOMETRIC FUNCTIONS

Angular variables are often denoted by Greek letters, such as θ or ϕ. We will use θ. *The angle should be measured in radians.* See Figure fig:radians ??. If s is the arclength of a section of a circle with radius r, then the angle of the section, in radians, is $\theta = s/r$. For example, the angle of a semicircular arc is $\theta = \pi r/r = \pi$ radians, because the arclength of the half circle is πr. This angle is 180 degrees, so to convert between radians and degrees we use the equivalence

$$\pi \text{ radians } = 180 \text{ degrees.}$$

For example, a *right angle* is 90 degrees or $\pi/2$ radians.

The well-known trigonometric functions are $\sin\theta$, $\cos\theta$, and $\tan\theta$. These are defined from the lengths in a right triangle in Figure fig:rightT ??. Two important identities of the trig functions can be derived from the right-triangle formulae. First, note that

$$\frac{\sin\theta}{\cos\theta} = \frac{O/H}{A/H} = \frac{O}{A} = \tan\theta \qquad \Longrightarrow \qquad \frac{\sin\theta}{\cos\theta} = \tan\theta. \tag{11-1}$$ eq:id1

Second, by the Pythagorean theorem,

$$\begin{aligned} & \sin^2\theta + \cos^2\theta = \frac{O^2 + A^2}{H^2} = \frac{H^2}{H^2} = 1 \\ \Longrightarrow \quad & \sin^2\theta + \cos^2\theta = 1. \end{aligned} \tag{11-2}$$ eq:id2

Less familiar trig functions are $\csc\theta$, $\sec\theta$, and $\cot\theta$, which are the *reciprocals*

$$\csc\theta = \frac{1}{\sin\theta}\ , \quad \sec\theta = \frac{1}{\cos\theta}\ , \quad \cot\theta = \frac{1}{\tan\theta}. \tag{11-3}$$

We'll analyze trig functions with θ as the independent variable. What are the domain and range? The trigonometric relations in Fig. fig:rightT ?? are limited to $0 < \theta < \pi/2$, because the angles in a right triangle must be less than 90 degrees. But we *extend* the definitions to angles greater than $\pi/2$ in Fig. fig:allangles ??. Note that the *sign* of each trig function is negative for some range of θ. Figure fig:gt2pi ?? shows the signs of the trig functions for angles greater than $\pi/2$. Because θ is an angle (in radians) it could be restricted to $[0, 2\pi]$. However, it is more useful to allow θ to take all real values, so that the domain is

$-\infty < \theta < \infty$. Because the angle $\theta + 2\pi$ corresponds to the same point P in the plane as θ, all the trig functions are *periodic* functions; in particular,

$$\sin(\theta + 2\pi) = \sin\theta, \tag{11-4}$$
$$\cos(\theta + 2\pi) = \cos\theta. \tag{11-5}$$

In graphical terms, a graph of $\sin\theta$ or $\cos\theta$ oscillates between $+1$ and -1. The curve repeats—taking exactly the same shape—over intervals of length 2π. For example, $\cos\theta$ goes from 1 to -1 and back to 1 as θ varies from 0 to 2π; then it repeats with the same shape as θ varies from 2π to 4π; and it repeats identically for any interval from $2\pi n$ to $2\pi(n+1)$ with n an integer. The graphs of $\sin\theta$ and $\cos\theta$ are shown in Fig. fig:sincos ??.[1] From the graphs we see that the range of either $\sin\theta$ or $\cos\theta$ is $[-1, 1]$. Also, these functions are continuous.

The function $\tan\theta$ has separate *branches*. A graph of $\tan\theta$ is shown in Fig. fig:tan ??.[2] The range of $\tan\theta$ is $(-\infty, +\infty)$. By the identity (eq:id1 11-1), $\tan\theta$ approaches $\pm\infty$ as θ approaches any value with $\cos\theta = 0$. The right-triangle formulae (see Fig. fig:rightT ??) show that $\cos(\pi/2) = 0$, because the adjacent length A tends to 0 as θ approaches $\pi/2$ (with H fixed). By periodicity, $\cos\theta$ is also 0 for $\theta = \pi/2 + n\pi$ for any integer n. Therefore the tangent is discontinuous, and undefined, at $\theta = \pi/2 + n\pi$. The period of the tangent function is π,

$$\tan(\theta + \pi) = \tan\theta. \tag{11-6}$$

Additional properties of the trigonometric functions are explored in the exercises.

[1] Please reproduce these graphs with a graphing calculator.
[2] Please reproduce the graph with a graphing calculator.

11.2 DERIVATIVES OF THE TRIG FUNCTIONS

The derivative of $\sin\theta$ is, by definition,

$$\frac{d}{d\theta}\sin\theta = \lim_{\delta\to 0}\frac{\sin(\theta+\delta)-\sin\theta}{\delta}. \qquad (11\text{-}7)$$ eq:defder

To evaluate the limit, simplify the right-hand side by applying the identity

$$\sin(A+B) = \sin A\cos B + \cos A\sin B; \qquad (11\text{-}8)$$

letting $A=\theta$ and $B=\delta$ gives

$$\sin(\theta+\delta) = \sin\theta\cos\delta + \cos\theta\sin\delta. \qquad (11\text{-}9)$$

Now take the limit $\delta \to 0$. The factor $\cos\delta$ may be approximated by 1, and $\sin\delta$ may be approximated by δ. (These crucial approximations are explained below.) Making these approximations, (eq:defder 11-7) becomes

$$\begin{aligned}\frac{d}{d\theta}\sin\theta &= \lim_{\delta\to 0}\frac{\sin\theta+\cos\theta\cdot\delta-\sin\theta}{\delta}\\ &= \lim_{\delta\to 0}\cos\theta = \cos\theta. \qquad (11\text{-}10)\end{aligned}$$

The result is

$$\frac{d}{d\theta}\sin\theta = \cos\theta. \qquad (11\text{-}11)$$ eq:dsinth

Before proceeding, let's make sure we understand the approximations

$$\cos\delta\approx 1 \qquad \text{and} \qquad \sin\delta\approx\delta, \qquad (11\text{-}12)$$

which are valid for small δ, i.e., $\delta \ll 1$. Figure fig:sincos ?? shows a graph of $\cos\theta$ and it is obvious from the graph that $\cos 0 = 1$, and $\cos\delta\approx 1$ for small δ. Figure fig:sincos ?? also shows $\sin\theta$. It is obvious from the graph that $\sin 0 = 0$, and $\sin\delta$ is approximately linear in δ for small δ. In fact, the slope of $\sin\theta$ is 1 at $\theta = 1$, so $\sin\delta\approx\delta$ for small δ. Figure fig:sinoverx ?? shows a graph of $\sin\theta/\theta$, and it is obvious that $\sin\theta/\theta \to 1$ as $\theta\to 0$; that is, $\sin\delta\approx\delta$ for small δ. But these analyses are merely numerical. A rigorous mathematical proof that $\sin\delta$ approaches δ as $\delta\to 0$ is given in Appendix X.

We have been denoting the independent variable by θ, because θ is a common notation for an angular variable in applications. But since the domain of the trig functions is $(-\infty,+\infty)$ we could just as well use the generic symbol x. Then the derivative formula (eq:dsinth 11-11) becomes

$$\frac{d}{dx}\sin x = \cos x. \qquad (11\text{-}13)$$ eq:dsin

$f(x)$	df/dx	$f(x)$	df/dx
$\sin x$	$\cos x$	$\csc x$	$-\cot x \csc x$
$\cos x$	$-\sin x$	$\sec x$	$\tan x \sec x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$

Table 11.1: Derivatives of the trigonometric functions tbl:dt

Next, what is the derivative of $\cos\theta$? We could go back to the basic definition,[3] but it is easier to use the fact that $\cos\theta$ is simply related to $\sin\theta$, by

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right). \tag{11-14}$$ eq:comple

For example, in the right triangle in Fig. fig:rightT ??, the side adjacent to θ is the side opposite to the complementary angle $\psi \equiv \pi/2 - \theta$; so

$$\cos\theta = \frac{A}{H} \qquad \text{and} \qquad \sin\psi = \frac{A}{H},$$

which implies (eq:comple 11-14). [Similarly, $\sin\theta = \cos(\pi/2 - \theta)$.] Now, to differentiate $\cos\theta$, apply the chain rule to the identity (eq:comple 11-14),

$$\begin{aligned} \frac{d}{d\theta}\cos\theta &= \frac{d}{d\theta}\sin\left(\frac{\pi}{2} - \theta\right) \\ &= \frac{d}{du}(\sin u) \times \frac{du}{d\theta} \qquad \text{where} \qquad u = \frac{\pi}{2} - \theta \\ &= \cos u \times (-1) = -\cos(\pi/2 - \theta) = -\sin\theta. \end{aligned} \tag{11-15}$$

Hence the derivative of the cosine function is

$$\frac{d}{dx}\cos x = -\sin x. \tag{11-16}$$ eq:dcos

Table tbl:dt 11.1 records the derivatives of the trigonometric functions, starting with the sine and cosine functions.[4] The derivatives of the other functions can be derived by first expressing the function in terms of sine and cosine, and then applying general methods of differentiation, as in the next two examples.

[3] See Exercise exer:dcos ??.

[4] The derivatives of trigonometric functions are also listed in the Table of Derivatives in Appendix E.

Example 1. Determine the derivative of $\tan x$.

Solution. Recall the identity (eq:id1 11-1),

$$\tan x = \frac{\sin x}{\cos x} = \frac{N(x)}{D(x)}. \tag{11-17}$$

Calculate the derivative by using the rule for differentiating a quotient,

$$\begin{aligned} \frac{d}{dx}\tan x &= \frac{N'D - D'N}{D^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned} \tag{11-18}$$

Example 2. Determine the derivative of $\sec x$.

Solution. Recall that $\sec x = 1/\cos x$. Therefore

$$\begin{aligned} \frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} = \frac{-1}{\cos^2 x}\frac{d}{dx}\cos x \\ &= \frac{\sin x}{\cos^2 x} = \tan x \, \sec x. \end{aligned} \tag{11-19}$$

Rather than try to memorize all the trig derivatives, it is sufficient to know (eq:dsin 11-13) and (eq:dcos 11-16). Then the other derivatives can be determined when needed in the manner of Examples 1 and 2.

An interesting application of the derivatives of $\sin\theta$ and $\cos\theta$ occurs in the use of plane polar coordinates to describe motion in 2 dimensions. See Appendix P.

11.3 TAYLOR SERIES FOR TRIGONOMETRIC FUNCTIONS

The functions $\cos x$ and $\sin x$ may be written as power series in x. These expansions reveal a beautiful relation between trig functions and the exponential function e^x. We shall see that the exponential of an *imaginary* variable $i\theta$ (where $i = \sqrt{-1}$) is a combination of $\cos\theta$ and $\sin\theta$. This is not merely a wonderful but abstract relationship; it leads to important practical methods of analysis, such as the Laplace transform.

Example 3. Derive the Taylor series expansion for $\cos x$, around $x = 0$. (In other words, the result will be the Maclaurin series for $\cos x$.)

Solution. Let $f(x) = \cos x$. The Taylor series expansion [recall Eq. (7-xx) or (7-yy)] around $x = 0$ is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2!}f^{(2)}(0)x^2 \\ &\quad + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots \end{aligned} \tag{11-20}$$

eq:Tay

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. The first few derivatives are shown in the table[5]

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1

A simple pattern emerges, which repeats after every 4 steps. Note that the fourth derivative $f^{(4)}(x)$ is the same as $f(x)$, namely $\cos x$, which is where we started. So the next four derivatives are again $\{-\sin x, -\cos x, \sin x, \cos x\}$ and the pattern starts over again at $n = 8$. Thus

$$f^{(n)}(x) = \begin{cases} \cos x & \text{if} \quad n = 4\nu \quad \text{(divisible by 4)} \\ -\sin x & \text{if} \quad n = 4\nu + 1 \\ -\cos x & \text{if} \quad n = 4\nu + 2 \\ \sin x & \text{if} \quad n = 4\nu + 3 \end{cases} \tag{11-21}$$

[5]Please verify the table, from the relations $\sin' = \cos$ and $\cos' = -\sin$.

where ν is any integer. The derivatives evaluated at $x = 0$ are

$$f^{(n)}(0) = \begin{cases} +1 & \text{if} \quad n = 4\nu \\ -1 & \text{if} \quad n = 4\nu + 2 \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{11-22}$$

Inserting these results into (11-20), the Taylor series is

$$\begin{aligned} \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}. \end{aligned} \tag{11-23}$$

eq:TScos

The Taylor series of $\sin x$ can be derived by a similar calculation[6] and the result is

$$\begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \end{aligned} \tag{11-24}$$

eq:TSsin

The series in (11-23) and (11-24) converge for any value of x.

⋆ ⋆ ⋆

The Taylor series for the other trigonometric functions are more complicated. Consider the tangent function, $g(x) = \tan x$. The first three nonzero terms in the Taylor series are

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots, \tag{11-25}$$

and the full series is complicated and has no particular interest. However, it will turn out that the *inverse* function, $\arctan x$, has a simple and interesting Taylor series.

11.3.1 Cosine, sine, and exponential

The cosine and sine functions originate in trigonometry—the analysis of plane triangles. This branch of mathematics is necessary for practical applications such as surveying, civil engineering, and statics and dynamics of mechanical systems. Through calculus we discover something else about the cosine and sine functions: They are related to the exponential function with a complex variable.[7]

[6] Exercise ??.

[7] The complex numbers are numbers that have both real and imaginary components, $x + iy$ where $i = \sqrt{-1}$.

In general any complex number z may be written as

$$z = x + iy$$

where x and y are real,

$$x = \operatorname{Re} z \qquad \text{and} \qquad y = \operatorname{Im} z.$$

The exponential function e^z is defined for a complex variable z through the power series,

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!}. \end{aligned} \qquad (11\text{-}26)$$

eq:psexp

If z is real, $z = x$, then (11-26) is the Taylor series expansion for e^x, which was derived in Chap. 10.

If z is imaginary, $z = iy$ with real y, then (11-26) separates into the trigonometric series of cosine and sine. Writing the first few terms of the series in (11-26), with $z = iy$,

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \frac{y^6}{6!} - \cdots \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots + i\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots \right]. \end{aligned} \qquad (11\text{-}27)$$

eq:deriveEuler

All terms with an even power of y are real, and those with an odd power of y are imaginary. The signs of the various terms are determined by the powers of i,

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad etc. \qquad (11\text{-}28)$$

Any integer power of i must be one of the set $\{1, i, -1, -i\}$. If n is divisible by 4 then $i^n = 1$; considering all integer powers,

$$i^n = \begin{cases} +1 & \text{if} \quad n = 4\nu \text{(divisible by 4)} \\ +i & \text{if} \quad n = 4\nu + 1 \\ -1 & \text{if} \quad n = 4\nu + 2 \\ -i & \text{if} \quad n = 4\nu + 3 \end{cases} \qquad (11\text{-}29)$$

where ν is any integer. Now examine the real and imaginary parts of (11-27). These are just the series (11-23) and (11-24), respectively, for $\cos y$ and $\sin y$. Hence

$$e^{iy} = \cos y + i \sin y. \qquad (11\text{-}30)$$

eq:Euler

This identity is called Euler's relation. It shows that the exponential and trigonometric functions are intimately related, through *complex numbers*.

Equation (11-30) expresses the exponential in terms of cosine and sine,

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{11-31}$$

eq:Eulertheta

Conversely, we may express cosines and sines in terms of exponential functions. First note that

$$\cos(-\theta) = \cos\theta \qquad \text{and} \qquad \sin(-\theta) = -\sin\theta. \tag{11-32}$$

Then by (11-31)

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{11-33}$$

eq:Eulernegth

Adding (11-31) and (11-33) the $i\sin\theta$ terms cancel and we obtain the cosine,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{11-34}$$

eq:cosexp

Subtracting (11-33) from (11-31) we obtain the sine,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{11-35}$$

eq:sinexp

In deriving (11-34) and (11-35) we have explicitly assumed that θ is real, so that $i\theta$ is purely imaginary. But in fact all the calculations would be equally valid for complex numbers, and we may write in general

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{11-36}$$

eq:cosz

for any complex number z. Equations (11-36) may be taken as the definitions of trig functions for complex z, with the complex exponential defined by the power series (11-26).

11.4 THE INVERSE TRIGONOMETRIC FUNCTIONS

Inverse functions were introduced in Chapter 1. For example, consider the inverse of the sine function, denoted $\sin^{-1}$:

$$x = \sin\theta \qquad \Longleftrightarrow \qquad \theta = \sin^{-1} x. \tag{11-37}$$

Another notation for the inverse of sine is arcsin (read as "arc sine"),

$$\arcsin x = \sin^{-1} x. \tag{11-38}$$

The name, $\arcsin x$, is quite descriptive. $\text{Arcsin}\, x$ is the angle whose sine is x, because

$$\theta = \arcsin x \qquad \Longleftrightarrow \qquad x = \sin\theta. \tag{11-39}$$

To define an inverse function, the function must be one-to-one.[8] The function $\sin\theta$ is one-to-one for θ in $[-\pi/2, \pi/2]$; as θ varies from $-\pi/2$ to $\pi/2$, $\sin\theta$ varies from -1 to $+1$. Therefore the domain of $\arcsin x$ is $x \in [-1, 1]$, and the range is $[-\pi/2, \pi/2]$. Figure fig:arcsin ?? shows a graph of $\arcsin x$.

To determine the derivative of $\arcsin x$ we need the relation of differentials. Let $\theta = \arcsin x$; then $x = \sin\theta$. From Sec. 11.2 we know the derivative $dx/d\theta = \cos\theta$; so the relation of differentials is

$$dx = \cos\theta\, d\theta. \tag{11-40}$$

In other words, if Δx is a small change of x and $\Delta\theta$ is the corresponding change of θ then

$$\Delta x \approx \cos\theta\, \Delta\theta. \tag{11-41}$$ eq:dsinis

The approximation here is that we neglect terms with a higher order of smallness than Δx or $\Delta\theta$; the higher-order terms are negligible in the limit $\Delta x \to 0$. But now we should express the ratio of changes as a function of x. For θ in $[-\pi/2, \pi/2]$ the trigonometric identity (eq:id2 11-2) determines $\cos\theta$,

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}. \tag{11-42}$$ eq:cosid

Combining the results (eq:dsinis 11-41) and (eq:cosid 11-42), we find the derivative of $\arcsin x$,

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \lim_{\Delta x \to 0} \frac{\Delta\theta}{\Delta x} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned} \tag{11-43}$$ eq:darcsin

[8] See Sec. 1.4.

Note that the derivative is consistent with the slope of the curve in Fig. fig:arcsin ??. The slope is positive for all x in $(-1,1)$, is equal to 1 for $x=0$, and tends to ∞ as x approaches ± 1. The function $1/\sqrt{1-x^2}$ has the same behavior.

The function $\arccos x$ is similar to $\arcsin x$, but has a different range. $\operatorname{Cos}\theta$ is a one-to-one function for θ from 0 to π (see Fig. fig:sincos ??). Therefore $\arccos x$ has domain $[-1,1]$ and range $[0,\pi]$. The relation of differentials for $x=\cos\theta$ is

$$dx = -\sin\theta\, d\theta. \tag{11-44}$$

So the derivative of $\theta = \arccos x$ is

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{-1}{\sin\theta} = \frac{-1}{\sqrt{1-\cos^2\theta}}, \\ \frac{d}{dx}\arccos x &= \frac{-1}{\sqrt{1-x^2}}. \end{aligned} \tag{11-45}$$

eq:darccos

The slope of $\arccos x$ is negative.[9]

Example 4. Differentiate $g(\xi) = \arctan\xi$.

Solution. The function $\xi = \tan g$ is one-to-one for $-\pi/2 < g < \pi/2$; this is evident from Fig. fig:tan ??. The corresponding range is $-\infty < \xi < \infty$. Therefore the domain of $\arctan\xi$ is $(-\infty,\infty)$ and the range is $(-\pi/2,\pi/2)$. A graph of $\arctan\xi$ is shown in Fig. fig:arctan ??. The relation of differentials is

$$d\xi = \left(\frac{d\tan g}{dg}\right) dg = \sec^2 g\, dg, \tag{11-46}$$

so the derivative of $\arctan\xi$ is

$$\frac{dg}{d\xi} = \frac{1}{\sec^2 g} = \cos^2 g. \tag{11-47}$$

But we must re-express the result as a function of ξ. Given that $\xi = \tan g$, what is $\cos g$? Recalling the identities (eq:id1 11-1) and (eq:id2 11-2),

$$\xi = \frac{\sin g}{\cos g} = \frac{\sqrt{1-\cos^2 g}}{\cos g}. \tag{11-48}$$

This may be solved for $\cos g$ in terms of ξ; after a bit of algebra the result is

$$\cos^2 g = \frac{1}{\xi^2+1}. \tag{11-49}$$

[9] Exercise exer:dacos ??.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{x^2+1}$	$\operatorname{arccot} x$	$\dfrac{-1}{x^2+1}$
$\operatorname{arccsc} x$	$\dfrac{-1}{x\sqrt{x^2-1}}$	$\operatorname{arcsec} x$	$\dfrac{1}{x\sqrt{x^2-1}}$

Table 11.2: Derivatives of the inverse trigonometric functions. tbl:dinv

Hence the derivative of $\arctan\xi$ is

$$\frac{d}{d\xi}\arctan\xi = \frac{1}{\xi^2+1}. \tag{11-50}$$ eq:darctan

Note that the derivative in (eq:darctan 11-50) agrees with the slope of $\arctan\xi$ in Fig. fig:arctan ??. The slope is maximum at $\xi = 0$ and approaches 0 as $\xi \to \pm\infty$.[10]

Table tbl:dinv 11.2 lists the derivatives of all the inverse trigonometric functions.

11.4.1 The Taylor series for $\arctan x$

Something interesting occurs in differentiating the inverse trig functions. Consider, for example, $F(x) = \arctan x$. We start with a transcendental function. But $F'(x)$ is merely algebraic,

$$F(x) = \arctan x \qquad \text{and} \qquad F'(x) = \frac{1}{x^2+1}. \tag{11-51}$$

Now all the higher derivatives will also be algebraic,

$$F^{(2)}(x) = \frac{-2x}{(x^2+1)^2}\ , \quad F^{(3)}(x) = \frac{6x^2-2x}{(x^2+1)^3}\ , \quad \text{etc.} \tag{11-52}$$

A very beautiful result that we may now derive is the Maclaurin series for $\arctan x$, i.e., the Taylor series around $x = 0$. We'll start with $F'(x)$, for which the Maclaurin series can be written by inspection,

$$F'(x) = 1 - x^2 + x^4 - x^6 + x^8 - + \cdots \tag{11-53}$$ eq:serFp

i.e., simply the geometric series for $1/(x^2+1)$. Now, what is the function $F(x)$ whose derivative is (eq:serFp 11-53)? Each term on the right side of (eq:serFp 11-53) is a

[10] Exercise exer:datan ??.

power $\pm x^p$; and the function with derivative $\pm x^p$ is $\pm x^{p+1}/(p+1)$. Therefore,

$$F(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - + \cdots \ . \qquad (11\text{-}54)$$ eq:arctanTS

Thus $\arctan x$ has a simple Taylor series. A striking example of the series occurs for $x = 1$. note that $F(1) = \arctan 1 = \pi/4$, and hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \cdots \ . \qquad (11\text{-}55)$$ eq:Leibnizseries

In words, *the alternating series of reciprocal odd integers is equal to* $\pi/4$. This amazing result was published by Leibniz in 1674, and known earlier to James Gregory. It shows that the transcendental number π is in fact *a very simple number*: it can be expressed using each odd integer exactly once.

11.5 THE HARMONIC OSCILLATOR

A general physics problem, with extensive applications in both science and engineering, is to describe *vibrations*. The simplest example is the harmonic oscillator, illustrated in Fig. fig:ho ??. We shall see that sinusoidal functions ($\cos\theta$ or $\sin\theta$) describe the motion.

Imagine a mass m that moves along the x axis, always attracted to the origin by a force proportional to the distance from the origin,

$$F = -kx. \tag{11-56}$$

The positive constant k is called Hooke's constant.[11] If x is positive then F is negative (i.e., the force is to the left, toward the origin); if x is negative then F is positive (i.e., the force is to the right, also toward the origin). Such a force is called a *restoring force* because it always acts in the direction toward the equilibrium position, $x = 0$. The fact that the restoring force for an elastic spring is proportional to the displacement from equilibrium is called Hooke's law. The linear relation, *force* $\propto -$ *displacement*, is characteristic of all kinds of small vibrations.

The equation of motion for the position $x(t)$ of m as a function of time t is

$$m\frac{d^2x}{dt^2} = -kx(t); \qquad \text{(Newton's second law)} \tag{11-57}$$

eq:dyn1

or,

$$\frac{d^2x}{dt^2} = -\omega^2 x(t) \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{m}}. \tag{11-58}$$

eq:dyn2

The general solution of (eq:dyn2 11-58) requires a function whose *second derivative* is the opposite of itself. Now think about $\cos\theta$.[12] Its derivative is $-\sin\theta$, and the derivative of that is $-\cos\theta$. So the second derivative of $\cos\theta$ is $-\cos\theta$. This functional property is just what we are looking for! We can also multiply by a constant A. Also, the variable θ should be linear in t and dimensionless, which suggests $\theta = \omega t - \delta$. So, we are led to try the general solution

$$x(t) = A\cos(\omega t - \delta). \tag{11-59}$$

eq:general

[11] Robert Hooke (1635 – 1703) was a contemporary of Isaac Newton. The two men had a rather unfriendly relationship due to controversies over their relative merit in science.

[12] We could alternatively use $\sin\theta$ to describe the oscillating position. The functions $\cos\theta$ and $\sin\theta$ are really equivalent for this purpose, because they differ only by a phase shift.

Indeed this function does obey the differential equation (eq:dyn2 11-58):

$$\frac{dx}{dt} = -\omega A \sin(\omega t - \delta)$$
$$\frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t - \delta) = -\omega^2 x.$$

There are two constant parameters, A and δ, in the general solution. These will be determined if some initial conditions of the system are specified.

Example 5. Suppose the mass m in Fig. fig:ho ?? is pulled to the position $x = 3\,\mathrm{cm}$, held at rest at that point, and then released at time $t = 0$. Determine x as a function of t.

Solution. Two initial conditions are given. The position at $t = 0$ is $x(0) = 3\,\mathrm{cm}$. The velocity at $t = 0$ is 0 (i.e., m is released from rest) so $x'(0) = 0$. Applying these conditions to the general solution (eq:general 11-59) we obtain two equations for the two unknown parameters, A and δ,

$$x(0) = A\cos\delta = 3\,\mathrm{cm} \tag{11-60}$$ eq:IC1
$$x'(0) = \omega A \sin\delta = 0. \tag{11-61}$$ eq:IC2

The second equation is satisfied by $\delta = 0$. Then the first equation implies $A = 3\,\mathrm{cm}$. The result is

$$x(t) = (3\,\mathrm{cm})\,\cos\omega t. \tag{11-62}$$

Figure fig:hograph ?? shows a graph of x versus ωt. Note that the slope of x versus t, which is the *velocity*, is 0 at $t = 0$, as required. The mass m oscillates back and forth between $+3\,\mathrm{cm}$ and $-3\,\mathrm{cm}$ with a period of $T = 2\pi/\omega$. For example, at time $t = T$ the mass is again at the initial point and instantaneously at rest.

Example 6. If the period of the harmonic oscillator in Fig. fig:ho ?? is $0.5\,\mathrm{s}$, and the oscillating mass is $150\,\mathrm{g}$, what is Hooke's constant for the spring, in N/m?

Solution. The period of oscillation is

$$T = \frac{2\pi}{\omega} = 0.5\,\mathrm{s}, \tag{11-63}$$

and note that $\omega = 2\pi/T$. But ω is defined in (eq:dyn2 11-58), which implies $k = m\omega^2$. Thus Hooke's constant is

$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 \times 150 \times 10^{-3}\,\mathrm{kg}}{(0.5\,\mathrm{s})^2}$$
$$= 2.4\,\mathrm{N/m}. \tag{11-64}$$

(The newton N is $1\,\mathrm{kg\,m/s^2}$.)

Example 7. Let ω be $12\,\mathrm{s}^{-1}$. Suppose the mass m is initially (at $t = 0$) at position $x = 3\,\mathrm{cm}$, and is given an initial velocity (toward larger x) of $24\,\mathrm{cm/s}$. At what time will m first pass the equilibrium point? What is the instantaneous velocity at that time?

Solution. We will need the parameters A and δ in the general solution (11-59). The initial conditions in this case are

$$\begin{aligned} x(0) &= A\cos\delta = 3\,\mathrm{cm}, \\ x'(0) &= \omega A \sin\delta = 24\,\mathrm{cm/s} \qquad \Rightarrow \qquad A\sin\delta = 2\,\mathrm{cm}. \end{aligned} \tag{11-65}$$

eq:eqsforpars

We may solve for A by adding the squares,

$$\begin{aligned} A^2 &= (A\cos\delta)^2 + (A\sin\delta)^2 = 13\,\mathrm{cm}^2 \\ A &= 3.61\,\mathrm{cm}; \end{aligned} \tag{11-66}$$

and then solve the second equation in (11-65) for δ,

$$\delta = \arcsin\left(\frac{2}{3.61}\right) = 0.587. \tag{11-67}$$

The position as a function of time is $A\cos(\omega t - \delta)$. The time t_0 when m passes the origin ($x = 0$) is given by

$$\omega t_0 - \delta = \frac{\pi}{2} \qquad \Rightarrow \qquad t_0 = \frac{1}{\omega}\left(\frac{\pi}{2} + \delta\right) = 0.18\,\mathrm{s}. \tag{11-68}$$

The velocity at that time is

$$x'(t_0) = -\omega A \sin(\omega t_0 - \delta) = -\omega A = -43.3\,\mathrm{cm/s}. \tag{11-69}$$

The negative velocity indicates that m is moving toward smaller x, i.e., in the negative x direction, as m passes the origin at time t_0.

11.5.1 Kinetic and Potential Energy

The kinetic energy of a mass m moving with velocity v is $K = \frac{1}{2}mv^2$. The potential energy of a spring k extended or compressed by length x is $U = \frac{1}{2}kx^2$. The total energy of the harmonic oscillator in Fig. ?? is a constant of the motion.

The energies of the harmonic oscillator, for the general solution (11-59), are

$$K(t) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta), \tag{11-70}$$

$$U(t) \quad = \quad \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta). \tag{11-71}$$

Then the total energy is

$$E = K(t) + U(t) = \tfrac{1}{2}kA^2 \tag{11-72}$$

where $m\omega^2$ has been replaced by k in accord with (eq:dyn2 11-58), and the trigonometric identity (eq:id2 11-2) has been used. The energy is a constant of the motion.[13]

Conservation of energy is one of the great unifying principles in science. Its simplest expression is in mechanics, where kinetic energy plus potential energy is constant. The harmonic oscillator is a basic example of this principle.

⋆

The system in Fig. fig:ho ?? is very simple. We have ignored friction or other damping forces, so the oscillations would continue forever with constant amplitude. Although this is an idealized example, it is important because all vibrating systems with small amplitudes resemble the harmonic oscillator mathematically, over times small enough that damping is negligible. In particular, the sinusoidal functional form of the oscillating variable is common.

Why does the behavior of *vibrations* depend on a trig function, $\cos\theta$ or $\sin\theta$? After all, the physical problem has nothing to do with right triangles—the trigonometric origin of the sine and cosine. But nature is simple, reusing the same functions for different applications. It is because of *calculus* that harmonic oscillations are sinusoidal. The dynamical equations of a harmonic oscillator are

$$\frac{dx}{dt} = v \qquad \text{and} \qquad \frac{dv}{dt} = -\omega^2 x. \tag{11-73}$$

These are *dual relations* between the position $x(t)$ and velocity $v(t)$. Note the resemblance to the derivatives of $f(\theta) \equiv \sin\theta$ and $g(\theta) \equiv \cos\theta$,

$$\frac{df}{d\theta} = g \qquad \text{and} \qquad \frac{dg}{d\theta} = -f. \tag{11-74}$$

Only the sine and cosine functions have these dual relations of derivatives, so these functions are key to the mathematics of vibrations.

[13] Exercises exer:const1 ?? and exer:const2 ??.

11.5.2 Harmonic waves

Harmonic oscillations, and the sinusoidal functions, also occur in waves. But the function for a wave depends on multiple variables—spatial coordinates and time.

In general, what is a wave?

A wave is a geometrical structure that is extended in space, and that oscillates in both space and time. Many physical phenomena involve waves: sound, light, motion of a violin string, water height in the ocean, etc. In any harmonic wave, some quantity varies sinusoidally, both as a function of position for fixed time and as a function of time for fixed position. Examples of oscillating quantities in various wave phenomena are listed in the table:

phenomenon	oscillating quantities
sound	pressure and density
light	electric and magnetic fields
violin music	transverse displacements
	of the string and box
ocean wave	vertical displacement
	of the water surface

Figure fig:wave ?? illustrates a general wave motion. The wave moves in the x direction, and $Q(x,t)$ is the quantity that varies in the wave. Figure fig:wave ?? shows a *snapshot* of the wave—a graph of Q versus x at an instant of time. Now imagine what happens as the wave moves to the right. The shape remains sinusoidal, but the positions of crests, troughs, and nodes travel along the x axis. An observer at a fixed point will find Q varying in time as the wave moves past the observer's position. For example, suppose Jack is at the origin and Jill is one-half wavelength to the right. Both observe $Q = 0$ at the time in Fig. fig:wave ??. But as the wave moves (to the right) Q decreases for Jack (becoming negative) and increases for Jill (becoming positive) until the wave has moved one-quarter wavelength. At that time, the trough (minimum Q) is by Jack while the crest (maximum Q) is by Jill. As time passes, Q oscillates for both Jack and Jill.

The function $Q(x,t)$ that describes a harmonic wave moving in one dimension (x) is

$$Q(x,t) = A\cos\left(kx - \omega t - \delta\right). \qquad (11\text{-}75)$$ eq:wavefun

There are four parameters. A is the *amplitude*, i.e., the maximum size of Q; Q oscillates between $-A$ and $+A$. δ is the *phase shift*, which determines the positions and times for wave crests, troughs, and nodes. k and ω are related to the wavelength and frequency. $\text{Cos}\,\theta$ is a periodic function of θ, with perion 2π,

$$\cos(\theta + 2\pi) = \cos\theta.$$

Therefore $Q(x, t)$ has the same value at x and $x + \lambda$ (at the same time t) where $k\lambda = 2\pi$; the wavelength is $\lambda = 2\pi/k$. Similarly, $Q(x, t)$ has the same value at t and $t + \tau$ (at the same location x) where $\omega\tau = 2\pi$; the period of oscillation is $\tau = 2\pi/\omega$ and the frequency is $f = 1/\tau = \omega/2\pi$.

Throughout science and engineering, waves are described by sinusoidal functions, $\sin\theta$ or $\cos\theta$.

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