The Exponential Function

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The exponential function is an important function in applied mathematics. Every scientific calculator has an e^x button for evaluating the exponential function. Today we'll explore some properties and applications of this function.

Denote the exponential function by $E(x)$. That is,

$$E(x) = e^x \tag{1}$$

where e is the base of natural logarithms, $e = 2.718...$. If $y = e^x$ then $x = \ln y = \log_e(y)$. That is, E and ln are inverse functions: $E[\ln(y)] = y$ and $\ln[E(x)] = x$.

A. Theorem 1

If x_1 and x_2 are any two numbers, then

$$E(x_1)E(x_2) = E(x_1 + x_2) \; ; \tag{2}$$

or,

$$e^{x_1}\, e^{x_2} = e^{x_1+x_2} \; . \tag{3}$$

It's easy to prove this theorem if x_1 and x_2 are integers n_1 and n_2:

$$\begin{aligned} e^{n_1} &= eee\cdots e \quad (n_1 \text{ times}) \\ e^{n_2} &= eee\cdots e \quad (n_2 \text{ times}) \\ e^{n_1}e^{n_2} &= eee\cdots e \quad (n_1 + n_2 \text{ times}) \\ &= e^{n_1+n_2} \end{aligned}$$

(It's fairly easy to prove Eq. (3) if x_1 and x_2 are rational numbers, but not so obvious if they are irrational.)

<u>Exercise 1</u>
Verify Eq. (2) for $x_1 = 1$ and $x_2 = 2$. Calculate $E(1)$, $E(2)$, $E(1)E(2)$, and $E(3)$.

However, Eq. (2) is not a unique property of the exponential function. The same equation holds for any base number. For example,

$$10^{x_1}\, 10^{x_2} = 10^{x_1+x_2} \; .$$

<u>Exercise 2</u>
Calculate 10^1, 10^2, and $10^1 \times 10^2$.

Or, generally where a is any number,

$$a^{x_1}\, a^{x_2} = a^{x_1+x_2} \; .$$

So what's so special about $E(x)$ and e? We'll return to this question after 2 examples.

B. Example 1: Compound Interest

Suppose you put \$100 in the bank on January 1, 2002, and the bank agrees to add 5% of the current value every January 1, in 2003, 2004, 2005, ... How much money will be in the account after January 1, 2102?

Let S_n be the amount in the account in year n. (Year 0 is 2002, year 1 is 2003, year 2 is 2004, etc.) Then

$$\begin{aligned} S_0 &= 100.00 \\ S_1 &= S_0 + 0.05\, S_0 = 1.05\, S_0 = 105.00 \\ S_2 &= S_1 + 0.05\, S_1 = (1.05)^2\, S_0 = 110.25 \\ S_3 &= S_2 + 0.05\, S_2 = (1.05)^3\, S_0 = 115.76 \end{aligned}$$

I hope you see the pattern,

$$S_n = S_{n-1} + 0.05\, S_{n-1} = (1.05)^n\, S_0 \; .$$

<u>Exercise 3</u>
Complete the following table.

year	n	S_n
2002	0	100.00
2003	1	105.00
2004	2	110.25
2005	3	115.76
2006	4	
2007	5	
2012	10	
2022	20	
2052	50	
2102	100	
3002	1000	

Now, what does compound interest have to do with $E(x)$? The growth of the sum of money for compound interest is an example of *exponential growth*. We can express the sum S_n in terms of the exponential function

$$S_n = (1.05)^n\; S_0 = e^{n \ln(1.05)}\; S_0 \; . \tag{4}$$

To prove the second equality in Eq. (4), note that any base number a to a power p can be expressed in terms of the exponential function

$$a^p = e^{p \ln a} = E(p \ln a) \ , \qquad (5)$$

because

$$e^{p \ln a} = e^{\ln a^p} = E\left[\ln(a^p)\right] = a^p \ .$$

C. Example 2: Radioactive Decay

The exponential function describes growth and decay. Compound interest is an example of growth. Radioactive decay is an example of exponential decay.

Suppose we start with N_0 radioactive atoms. We cannot say when any particular atom will decay. What we do know is the *half-life*, call it H, defined as follows: During the time interval H, one half of the atoms in a large sample will decay, and the other half will remain undecayed. Let $N(t)$ denote the number of radioactive atoms present at time t; then the number that *have not decayed* after additional time H is

$$N(t+H) = \frac{1}{2} N(t) \ .$$

Thus

$$\begin{aligned} N(H) &= \frac{1}{2}N(0) = \frac{1}{2}N_0 \\ N(2H) &= \frac{1}{2}N(H) = \left(\frac{1}{2}\right)^2 N_0 \\ N(3H) &= \frac{1}{2}N(2H) = \left(\frac{1}{2}\right)^3 N_0 \end{aligned}$$

I hope you see the pattern,

$$N(nH) = \left(\frac{1}{2}\right)^n N_0 \ .$$

Or, let $t = nH$ be an arbitrary time (not necessarily an integer multiple of H); then

$$N(t) = \left(\frac{1}{2}\right)^{t/H} N_0 \ . \qquad (6)$$

Again we may express $N(t)$ in terms of the exponential function. By Eq. (5), using the fact that $\ln(1/2) = -\ln(2)$,

$$N(t) = e^{-t \ln 2 / H} N_0 \ . \qquad (7)$$

Here the exponent is negative, so $N(t)$ decreases with t.

> Exercise 4
> The half-life of carbon-14 is $H = 5730\,\text{yr}$.
> (a) Plot the number of C-14 atoms $N(t)$ as a function of t, starting with 100 atoms at $t = 0$.
> (b) Carbon-14 is used in radioactive dating of organic matter. What is the age of a sample if the number of C-14 atoms is 10% of its original value, i.e., its value when the material was living matter?

D. The differential equation that defines $E(x)$

Theorem 1 does not define $E(x)$, because the same relation holds for any base number. So what does define $E(x)$? What is so special about the number e?

The exponential function describes growth or decay processes in which the rate of change of a quantity is proportional to the current amount of the quantity, as in Examples 1 and 2 above.

The rate of change of a function $f(x)$ is the *slope* of the curve in a graph of f versus x. We will define $E(x)$ by a property of its slope. But first, here is some background information on *the slope of a function.*

ON THE SLOPE OF A FUNCTION

The slope of a straight line, in a graph of y versus x, is the rise over the run, $\Delta y / \Delta x$. What about a curve that is not a straight line? Then the slope at a point on the curve is the slope of the tangent line at that point. We may calculate the slope at a point x, of the curve in a graph of a function $f(x)$, by a limit

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \qquad (8)$$

where $f'(x)$ denotes the slope of $f(x)$ at the point x. (See Figure 1.)

[In calculus, $f'(x)$ is called the derivative of $f(x)$.]

In Figure 1, the dashed line is the tangent line at $x = 1$. The slope of the dashed line is the same as the slope of $f(x)$ at $x = 1$.

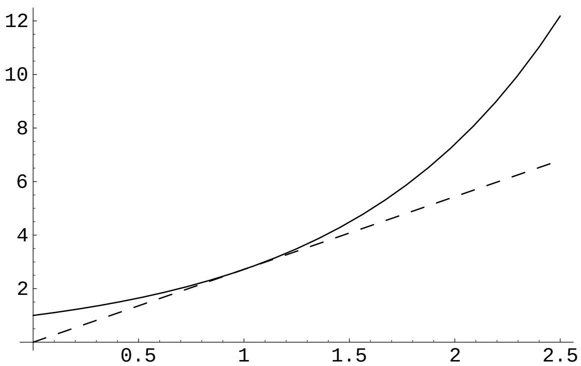

Figure 1. Determination of the slope of $f(x)$ at $x = 1$.

Definition of $E(x)$

The defining property of $E(x)$ is that *the slope of $E(x)$ at x is equal to the value of $E(x)$ at x*

$$E'(x) = E(x)\ . \tag{9}$$

Only the exponential function satisfies this equation with $E(0) = 1$.

Exercise 5
Let ϵ be small, say 1×10^{-6}. Estimate $E'(0)$, $E'(1)$, $E'(2)$ by calculating Eq. (8) (without the limit: just let $\epsilon = 10^{-6}$). Compare to $E(0)$, $E(1)$, $E(2)$, to 4 place accuracy.

x	$E'(x)$	$E(x)$
0		
1		
2		

E. Exponential growth is overwhelming.

The exponential function goes to infinity, as $x \to \infty$, *faster than any polynomial.* (A polynomial function of x is a sum of integer powers of x,

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n,$$

the sum having a finite number of terms.)

Exercise 6
(a) Plot $E(x)$ and $1 + x$ for x from 0 to 3, superimposed on the same graph.
(b) Compare $E(x)$ and the polynomial $1 + x^{10}$, for $x > 2$. When x is large enough, $E(x)$ is larger than $1+x^{10}$. How large must x be to have $E(x) > 1 + x^{10}$?

F. $E(x)$ as an infinite series

Since $E(x)$ grows faster than any polynomial, it cannot be expressed as a polynomial. Nevertheless, it can be written as a sum of integer powers of x, but with an infinite number of terms in the sum. Specifically,

$$E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \tag{10}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{11}$$

In calculus, the sum of an infinite number of terms is called an infinite series.

Are you surprised that the sum of an infinite number of positive numbers can be finite?

Exercise 7
Add the first 7 terms of Eq. (10) for $x = 1$, and compare the result to $E(1)$. Use 6 place accuracy.

Another example of an infinite series, remarkably related to the exponential function, is the infinite series for the cosine function of trigonometry,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - + \cdots \tag{12}$$

Exercise 8
Plot $\cos x$ and the sum of the first 5 terms in Eq. (12), for x from 0 to 5. (Superimpose the two functions on one graph.)

Proof of the exponential series (using calculus)

We need to prove that $E'(x) = E(x)$ for the series in Eq. (11), where $E'(x)$ denotes the derivative of $E(x)$. Recall that the derivative of x^p is px^{p-1}. Thus

$$\begin{aligned} E(x) &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ E'(x) &= 0 + \frac{1x^0}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}$$

$$= \quad E(x). \quad QED$$

G. $E(x)$ as a limit

Another expression for $E(x)$ is

$$E(x) = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N \quad ; \tag{13}$$

for example, for $x = 1$,

$$e = \lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^N \quad .$$

The limit as $N \to \infty$ is a delicate balancing act: The exponent N goes to infinity, tending to make the quantity blow up; but the base number $(1 + x/N)$ approaches 1, the multiplicative identity, tending to keep the quantity near 1. By these opposing tendencies the quantity remains finite as $N \to \infty$.

> Exercise 9
> (a) Plot $(1 + x/N)^N$ for $N = 10$, and $E(x)$, for x from 0 to 5. (Superimpose the two functions on one graph.)
> (b) Repeat for $N = 50$.

H. Example 3: The Equiangular Spiral

The equiangular spiral is the plane curve with the property that all radial rays (from the origin) intersect the spiral at the same angle α. The equation for an equiangular spiral, in plane polar coordinates (r, θ), is

$$r(\theta) = e^{\theta \cot \alpha} \quad . \tag{14}$$

Note that $r(\theta)$ is expressed in terms of the exponential function.

Snail shells have this shape. *Why do snails know about the exponential function?* The reason is that as the snail grows, its rate of growth is proportional to its size (small snail $\Rightarrow$ small growth rate, and big snail $\Rightarrow$ big growth rate); as we have learned, this is the property of exponential growth.

> Exercise 10
> Make a polar plot of $r(\theta)$ for $\alpha = 79$ degrees $= 1.37$ radians. Let the θ range be from -10 to 10. Prove that the curve is equiangular.

To make the snail plot with Mathematica:

```
Get["Graphics`Graphics`"]
const=Cot[79*Pi/180]
PolarPlot[Exp[const*theta],{theta,-10,10}]
```

Another method:

```
const=Cot[79*Pi/180]
r[th_]:=Exp[const*th]
ParametricPlot[{r[th]Cos[th],r[th]Sin[th]},
{th,-10,10}]
```

I. More fun: $E(z)$ for complex z

The exponential function is also defined as a function of complex numbers. Let z be the complex number $x + iy$ where x and y are real numbers, and $i = \sqrt{-1}$. Then by Theorem 1,

$$E(z) = E(x)E(iy) \ .$$

There is a connection between the exponential function and the trigonometric functions:

$$E(iy) = e^{iy} = \cos y + i \sin y \ ,$$

a result known as Euler's Theorem.

> Exercise 11 – extra credit
> Use Mathematica to make a 3D surface-graphics plot of the real part of $E(z)$ as a function of complex z, for x from -2 to 2 and y from -10 to 10.

```
Plot3D[Re[Exp[x+I*y]],{x,-2,2},{y,-10,10},
  PlotRange->{{-2,2},{-10,10},{-8,8}},
  PlotPoints->50]
```

Homework problems
due Thursday, Dec. 13

The assignment is to hand in solutions to any three of these problems. You choose which problems to do. For extra credit do more than three problems.

Problem 1. The limit of the sequence

$$\begin{aligned} & (1+1)^1, (1+1/2)^2, (1+1/3)^3, \ldots, \\ = \; & (1+1/n)^n \;, \quad \ldots \end{aligned}$$

is e. Let $\mathcal{E}_n$ denote the n^{th} term in this sequence. (For example, $\mathcal{E}_1 = (1+1)^1$, $\mathcal{E}_2 = (1+1/2)^2$, and in general $\mathcal{E}_n = (1+1/n)^n$.)

Calculate decimal values of $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_5$, $\mathcal{E}_{10}$, $\mathcal{E}_{100}$, and $\mathcal{E}_{1000}$, to 6 significant figures. How do they compare to e?

Problem 2. Equation (12), evaluated for $x = \pi$, implies

$$-1 = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - + \cdots, \tag{15}$$

a remarkable series! (There are many equations of this kind—relating π and the infinite set of integers.) Let M_n be the sum of the first n terms on the right-hand side of (15).

Calculate decimal values of M_1, M_2, M_3, $\cdots$, M_8, to six significant figures. How do they compare to -1?

Problem 3. Perhaps the most beautiful and amazing infinite series in mathematics is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \cdots, \tag{16}$$

proven by Leibniz in 1673. In words, the alternating series of reciprocal odd integers is $\pi/4$. This shows that π is a very simple number! It can be expressed using each odd integer exactly once.

Let Q_n be the sum of the first n terms on the right-hand side of (16). (For example, $Q_1 = 1$, $Q_2 = 2/3$, $Q_3 = 13/15$, etc.) Let $R_n = 2(Q_n + Q_{n+1})$.

(a) Calculate decimal values of R_1, R_2, R_3, R_5, R_{10}, to six significant figures.

(b) What is the limit of R_n as $n \to \infty$?

Problem 4. The Moebius strip

(a) Make a Moebius strip. Draw a line, half-way between the edges, around the circumference of the strip until it intersects itself; explain in words what happens. (This demonstration shows that the Moebius strip is a surface with only one side.)

(b) Use scissors to cut the strip around the circumference half-way between the edges, i.e., on the line you drew in (a). Explain in words what happens.

(c) Make another Moebius strip. Cut the strip around the circumference at a distance of one-third of the width from one edge, always keeping the scissors one third of the width from the nearest edge. Explain in words what happens.

The next two problems are ancient problems on geometric series. They illustrate that exponential growth is overwhelming.

Problem 5. Grains of wheat and chessboard

(This problem was published by Ibn Kallikan in 1256. You can look it up on the Internet if you want to.)

On a chessboard, 1 grain of wheat is placed on the first square, 2 on the second square, 4 on the third square, 8 on the fourth square, 16 on the fifth square, and so on for all 64 squares.

How many grains of wheat are needed?

Problem 6. Problem 79 from the Rhind Papyrus

(The Rhind Papyrus was written by the scribe Ahmes, who lived in Egypt from 1680 to 1620 BC. It was found by the archeologist Rhind in 1858.)

There are seven houses; in each house there are seven cats; each cat kills seven mice; each mouse has eaten seven grains of barley; each grain would have produced seven hekat [1]. What is the sum of all the enumerated things?

[1] Hekat = a volume measure of ancient Egypt

Answer Sheet **Nov 29**

Exercise 1
$E(1) =$ $E(2) =$
$E(1)E(2) =$
$E(3) =$

Exercise 3
Complete the table.

year	n	S_n
2002	0	100.00
2003	1	105.00
2004	2	110.25
2005	3	115.76
2006	4	
2007	5	
2012	10	
2022	20	
2052	50	
2102	100	

Exercise 4
(a) Sketch your plot of $N(t)$ versus t. Be sure to include a scale on the t axis.

(b) Age of the sample =

Exercise 5
Complete the table, using Eq. (8) (without the limit) to calculate $E'(x)$.

x	$E'(x)$	$E(x)$
0		
1		
2		

Exercise 6
(a) Sketch your plot of $E(x)$ and $1 + x$.

(b) How large must x be to have $E(x) > 1 + x^{10}$, for $x > 2$?

Exercise 7

Sum of the first seven terms =

value of $E(1)$ =

Exercise 8
Sketch your plot of $\cos x$ and the sum of five terms of the series in Eq. (12).

Exercise 9
(a) Sketch your plot of $(1 + x/10)^{10}$ and $E(x)$.

(b) Sketch your plot of $(1 + x/50)^{50}$ and $E(x)$.

Exercise 10
Sketch your plot of $r(\theta)$.