Convolution of plus distributions

Let $h(z)$ be the distribution

$$h(z) = [f_+ \otimes g_+] (z) = \int_z^1 \frac{dx}{x} f_+ \left( \frac{z}{x} \right) g_+(x).$$  \hfill (1)

The equation we need from Carl is

$$h(z) = \tilde{h}_+ (z) + A \delta (1 - z),$$  \hfill (2)

where $A$ is a constant (to be determined by comparing moments) and $\tilde{h}_+ (z)$ is the plus-distribution corresponding to a function $\tilde{h}(z)$ given by

$$\tilde{h}_+ (z) = \int_{z/a}^1 \frac{dx}{x} \left[ g_+ \left( \frac{z}{x} \right) - xg(z) \right] f(x)$$
$$+ \int_a^1 \frac{dx}{x} \left[ f_+ \left( \frac{z}{x} \right) - xf(z) \right] g(x)$$
$$- g(z) F(z/a) - f(z) G(a).$$  \hfill (3)

(There were some misprints in Carl’s notes.) Here

$$F(\xi) = \int_0^\xi f(x)dx \quad \text{and} \quad G(\xi) = \int_0^\xi g(x)dx.$$  \hfill (4)

The parameter $a$ is in the range $z < a < 1$, but all the $a$ dependence must cancel.

**Example 1**

1
Let \( f(x) = \frac{1}{1 - x} \) and \( g(x) = \frac{1}{1 - x} \).

This example is easy. The function \( \tilde{h}(z) \) is

\[
\tilde{h}(z) = \frac{1}{1 - z} \ln \left[ \frac{(1 - z)^2}{z} \right].
\]  \hspace{1cm} (5)

Then the distribution \( h(z) \) is

\[
h(z) = \left\{ \frac{1}{1 - z} \ln \left[ \frac{(1 - z)^2}{z} \right] \right\}_+. \]  \hspace{1cm} (6)

Note that the constant \( A \) in (2) is 0, which, I guess, is always true. One can verify that the moments are correct, using Mathematica.

**Example 2**

Let \( f(x) = \frac{1}{1 - x} \) and \( g(x) = \frac{\ln(1 - x)}{1 - x} \).

This example is harder. I can’t do it without Mathematica. The answer is a long formula with PolyLogs.

I more or less have this example finished, but the final result is a long formula. I did check that the result for \( \tilde{h}(z) \) is independent of \( a \). Then \( A = 0 \) in (2), and \( h(z) = \tilde{h}_+(z) \). I have not been able to verify that the moments are correct, because the integrals are hard even for Mathematica.

Dan