PHY 491: Atomic, Molecular, and Condensed Matter Physics
Michigan State University, Fall Semester 2012

Solve by: Wednesday, October 17, 2012

Homework 6 – Solution

6.1.

(i) Calculate the density of states of the electron gas in 2 and 1 dimensions.

(ii) Derive expressions for the Fermi energy in atomic units, where the energy is expressed in Hartree and the length is expressed in Bohr radius.

(iii) Consider a 2D electron gas with the density of 1.5×10^{11} cm^{-2}. Express this density in atomic units. What is the Fermi energy for this 2D electron gas?

Solution:

(i) In 2 dimensions (2D),

$$D(E)dE = 2 \cdot \frac{2\pi k\ dk}{\left(\frac{2\pi}{L}\right)^2} .$$

With

$$E = E_{\vec{k}} = \frac{\hbar^2 k^2}{2m} \Rightarrow k\ dk = \frac{m}{\hbar^2} dE$$

we get

$$D(E) = \frac{A}{2\pi}\left(\frac{2m}{\hbar^2}\right) ,$$

independent of E.

In 1 dimension (1D),

$$D(E)dE = 2 \cdot \frac{2\ dk}{\left(\frac{2\pi}{L}\right)}$$

with one factor of 2 for spin and the other factor of 2 for k and $-k$. With

$$k = \sqrt{\frac{2m}{\hbar^2}} E^{1/2} \Rightarrow dk = \frac{1}{2}\sqrt{\frac{2m}{\hbar^2}} E^{-1/2} dE$$

we get

$$D(E) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} E^{-1/2} .$$

(ii) In 2D, $E_F^* = N\pi\frac{1}{A^*}$, where $E_F^* = \frac{E_F}{\text{Hartree}}$ and $A^* = \frac{A}{a_B^2}$.

In 1D, $E_F^* = \frac{\pi^2}{8}\left(\frac{N}{L^*}\right)^2$, where $E_F^* = \frac{E_F}{\text{Hartree}}$ and $L^* = \frac{L}{a_B}$.

(iii)

$$\begin{aligned} \frac{N}{A} &= 1.5\times10^{11}\ \text{cm}^{-2} , \\ \frac{N}{A^*} &= \frac{N}{A/a_B^2} = \frac{N}{A} a_B^2 = 1.5\times10^{11}\ \text{cm}^{-2}\cdot(0.529\times10^{-8}\ \text{cm})^2 = 0.42\times10^{-5} , \end{aligned}$$

$$
\begin{aligned}
E_F^* &= \pi \frac{N}{A^*} = \pi \times 0.42 \times 10^{-5} \ , \\
E_F &= \pi \cdot 0.42 \times 10^{-5} \text{ Hartree} = \pi \cdot 0.42 \times 10^{-5} \times 27.2 \text{ eV} = 35.9 \times 10^{-5} \text{ eV} \ .
\end{aligned}
$$

6.2. The ^{3}He atom is a fermion with spin 1/2 (why?). The density of liquid ^{3}He is 0.081 g/cm^3 near $T = 0$. Calculate the Fermi energy E_F and the Fermi temperature T_F.

Solution:

Particles with a half-integer spin (here: nuclear spin) are fermions. The mass density of the liquid is

$$
\rho = \frac{N}{V} M(^3\text{He}) = 0.081 \ g/cm^3 \ .
$$

The number density is

$$
n = \frac{N}{V} = \frac{\rho}{M(^3\text{He})} = \frac{0.081 \ g/cm^3}{3 \times 1.67 \times 10^{-24} \ g} = 1.617 \times 10^{22} \ cm^{-3} = 1.617 \times 10^{28} \ m^{-3} \ .
$$

$$
\begin{aligned}
E_F &= \frac{\hbar^2}{2M(^3\text{He})} \left(3\pi^2 \frac{N}{V} \right)^{2/3} = 6.78 \times 10^{-23} \text{ J} = 4.24 \times 10^{-4} \text{ eV} \ , \\
T_F &= 4.24 \times 10^{-4} \times 1.16 \times 10^4 \text{ K} = 4.91 \text{ K} \ .
\end{aligned}
$$

6.3. Assuming a free electron gas model for the valence electrons of the metals Li, Na, Cs, Cu, Mg, Al, In, Pb, calculate the Fermi energy (in eV) and the zero point pressure (in atmospheric pressure). Use Table 4 *("Density and atomic concentration")* in Chapter 1 of Kittel (p. 24 in 7th edition, p. 21 in 8th edition).

Solution:

The expression for the zero point pressure derived in Class is

$$
P = \frac{2}{3} \left(\frac{3}{5} E_F \right) n \ ,
$$

where n is the electron density and E_F the Fermi level that can be expressed in terms of n. Make sure you get a result for a couple of elements (e.g. Na, Mg, In).
