Galileo: Inertial System

— If an object is moving with a constant velocity $\vec{v}$ (which contains magnitude (speed) ↳ v and direction).
vector "$\rightarrow$"

then it will remain in that velocity, when no external force.

"At rest" is for $\vec{v} = 0$.

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Newton: Three laws:

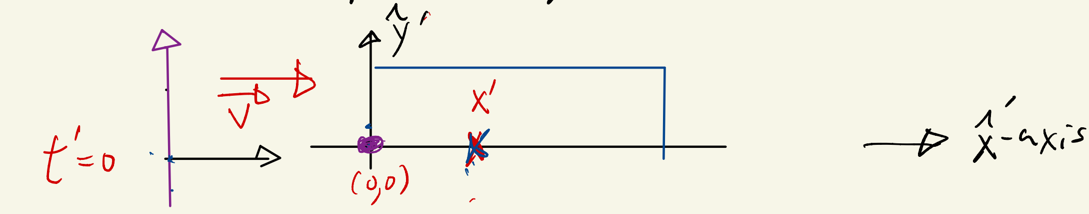
① Inertial system, when no force.
The law of physics (for describing object motion) is the same in all inertial systems.

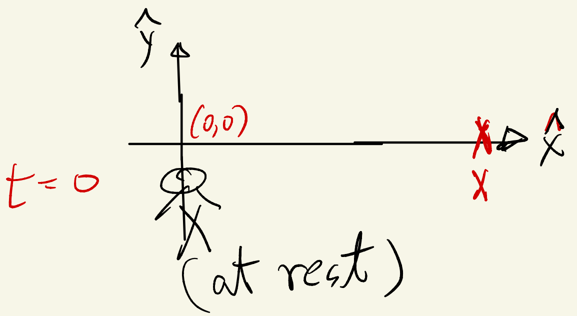
② With force, $\vec{F} = m\vec{a} = \frac{d(m\vec{v})}{dt}$

$$\left( \vec{v} = \frac{d\vec{x}}{dt} \right)$$

③ reaction = (action, but with opposite direction)

Consider two inertial systems.




(units:
$[v] = m/sec$
$[x] = m$
$[t] = sec$)

Newton: Time is absolute. $(t' = t)$

| | |
|---|---|
| $x' = x - vt$ | $x = x' + vt$ |
| $y' = y$ | $y = y'$ |
| $z' = z$ | $z = z'$ |
| $t' = t$ | $t = t'$ |

Galilean transformation

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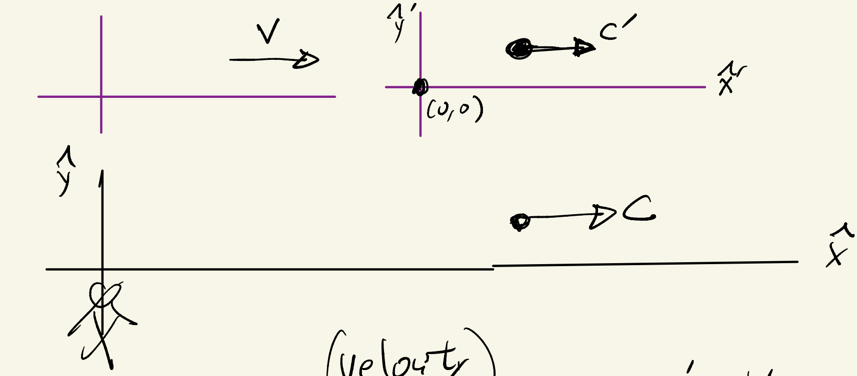
To describe an event, we need to specify both position and time, (and ...).

$(x, y, z, t)$




$c' = c - v$ (velocity) $c = c' + v$

What if $c$ and $c'$ are referring to the speed of Light?

$\Rightarrow$ $c \neq c'$ for $v \neq 0$

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Maxwell combined electricity & Magnetism.

$\Rightarrow$ Newton's law for describing classical mechanics (kinematics)
Maxwell's equations for describing E&M interactions

$\Rightarrow$ showed that all the electromagnetic waves move in vacuum with the same speed. $\leftarrow 3 \times 10^8 \, m/sec$

(Light is one of the EM wave.)

$\Rightarrow$ Light moves with the same speed (in vacuum) in all inertial systems.

Einstein: The principle of relativity:

Ⓐ All the physics laws (including classical mechanics and Electromagnetic interactions) should be the same in All the inertial systems.

$\Rightarrow$ There is no single absolute reference frame.

Ⓑ Speed of light (in vacuum) is the same in all inertial system.

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Consequences:

① $t' \neq t$, Time is Not absolute

② Time dilation

③ (Lorentz) length Contraction.

$\Rightarrow$ No speed of any object can exceed the speed of light

# How to see $t' \neq t$ (for $\vec{v} \neq 0$)?

① $x' = \gamma(x - vt)$ with ($\gamma \to 1$, as $v \to 0$)

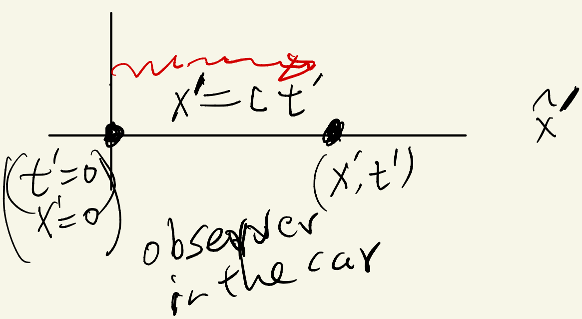
$x = \gamma'(x' + vt')$ with ($\gamma' \to 1$, as $v \to 0$)

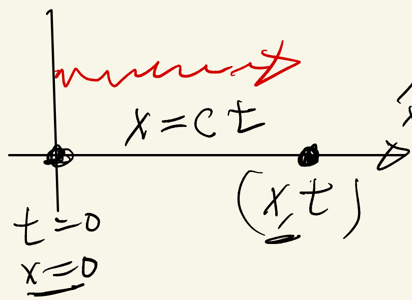
Based on A, we require $\boxed{\gamma' = \gamma}$. independent of inertial system. (reference frame)

② Based on B, we have

$x' = ct'$ and $x = ct$

the same

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Math is the natural language to describe Nature !

$$x' = \gamma(x - vt)$$

$$\cancel{ct'} = \gamma(ct - vt)$$

$$= \gamma(c-v)t$$

$$= ct\,\gamma\left(1 - \frac{v}{c}\right)$$

$$x = \gamma(x' + vt')$$

$$ct = \gamma(ct' + vt')$$

$$= \gamma(c+v)t'$$

$$= ct'\gamma\left(1+\frac{v}{c}\right)$$

$$= \cancel{ct'}\,\gamma\left(1+\frac{v}{c}\right)\gamma\left(1-\frac{v}{c}\right)$$

$$\Rightarrow 1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right) \Rightarrow \gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}}$$

($\gamma = 1$, when $v = 0$)

Note: $v$ cannot be larger than $c$, because of $\sqrt{1 - v^2/c^2}$.

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$x' = \gamma(x - vt)$, using $x' = ct'$ and $t = \frac{x}{c}$
$x = ct$

$$\cancel{c}t' = \gamma\left(ct - \frac{vx}{c}\right)$$

$$= \cancel{c}\gamma\left(t - \frac{vx}{c^2}\right)$$

$$\Rightarrow \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)},$$

(with $\gamma = 1$ and $t' = t$, when $v = 0$)

Relativistic kinematics is relevant when $v$ is close to the speed of light $c$.

# Transformation of coordinates

## Galileo-Newton

$$x' = x - v t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

$\underbrace{\qquad\qquad}$

$\vec{F} = m \cdot \vec{a}$ is invariant under these.

For $v \ll c$

$(\beta \approx 0, \gamma \approx 1)$

classical kinematics

## Lorentz-Fitzgerald

$$x' = \gamma (x - \beta c t)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma \left(t - \beta \frac{x}{c}\right)$$

$\beta = \frac{v}{c}$ : speed

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$ : gamma factor

$\underbrace{\qquad\qquad}$

The Maxwell equations are invariant under these, same is ~~for~~ $\vec{F} = \frac{d(m\vec{v})}{dt}$

For all $v$

$(v < c)$

Relativistic Kinematics

# Time dilation

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

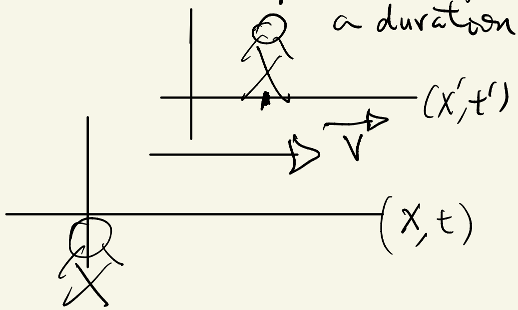
$$\Rightarrow \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \boxed{\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)}$$

light a candle for a duration $(\Delta t')$, with $(\Delta x' = 0)$




$$\boxed{(\Delta t) = \gamma(\Delta t')}$$

with $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 1$ for $v \neq 0$

when $v$ is close to $c$, say, $v = 0.99c$,

then $\gamma = \dfrac{1}{\sqrt{1-(0.99)^2}} \gg 1$

$\Rightarrow$ Time dilation.

# atmospheric muons

An atmospheric muon is one of the most common secondary particles produced when cosmic rays interact with Earth's atmosphere.

Primary cosmic rays are high-energy particles (mostly protons, plus some helium nuclei and heavier nuclei) that arrive from outer space.

When these primaries collide with nuclei in the upper atmosphere, they produce showers of secondary particles (pions, kaons, etc.).

These unstable mesons quickly decay, and one of the main decay products is the muon.

Because muons are relatively long-lived (2.2 microseconds at rest, extended by relativistic time dilation) and can penetrate matter deeply, they reach the ground in huge numbers. In fact, most of the cosmic radiation at Earth's surface comes from muons.

Cosmic ray (Atmospheric muons)

① muon is an elementary particle, similar to electon, but with mass 200 times of electron's.

$m_\mu = 200\, m_e = 0.1\ \mathrm{GeV}/c^2$, (GeV $= 10^9$ electron-Volt)

$c =$ speed of light $= 3\times10^8\ \mathrm{m/sec}$

② The energy of cosmic muon detected on earth is about 4 GeV.

③ the lifetime of muon has been measured to be $2.2\,\mu s = 2.2\times10^{-6}$ sec.

④ the height of the atmosphere of earth is about $15\ \mathrm{Km} = 15\times10^{3}\ \mathrm{m}$

(5) The speed of muon can be calculated by using

$\boxed{E = mc^2}$, with $\boxed{m = m_0 \gamma}$

$m_0$ = rest mass
= mass when at rest
($v=0$)

and

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$E = 4$ GeV
$m_0 = m_\mu = 0.1$ GeV$/c^2$

$$4 \cancel{\text{GeV}} = \left(0.1 \frac{\cancel{\text{GeV}}}{\cancel{c^2}}\right)\gamma \cdot \cancel{c^2}$$

$$\Rightarrow \gamma = \frac{4}{0.1} = 40$$

$$\Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2} = \frac{1}{(40)^2}$$

$$\Rightarrow 1-\frac{1}{(40)^2} = \frac{v^2}{c^2} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{(40)^2}}$$

$$\simeq 1 - \frac{1}{2}\frac{1}{(40)^2} + \cdots$$

$$\simeq 1 - \frac{1}{3200}$$

$$\simeq 0.999687\ldots$$

Wrong answers i.e., without time dilation,
(without special Relativity)

$\Rightarrow$ $(2.2\times10^{-6}\ \text{sec})\cdot(\text{speed of muon}) = (\text{distance that muon can travel})$

$\simeq (2.2\times10^{-6}\ \text{sec})\cdot(3\times10^{8}\ \text{m/sec})$

$= 6.6\times10^{2}\ \text{m}$

$\ll 15\times10^{3}\ \text{m}$

$\Rightarrow$ cosmic muon can never reach the Earth.

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Correct answer is that due to time dilation, the time duration seen by the observer on Earth is

$(2.2\times10^{-6}\ \text{sec})\cdot\gamma = (2.2\times10^{-6}\ \text{sec})\cdot(40)$

$= 8.8\times10^{-5}\ \text{sec}$

time dilation

$\Rightarrow$ The muon can travel

$(40)\cdot(6.6\times10^{2}\ \text{m}) = 26.4\times10^{3}\ \text{m}$

$> 15\times10^{3}\ \text{m}$

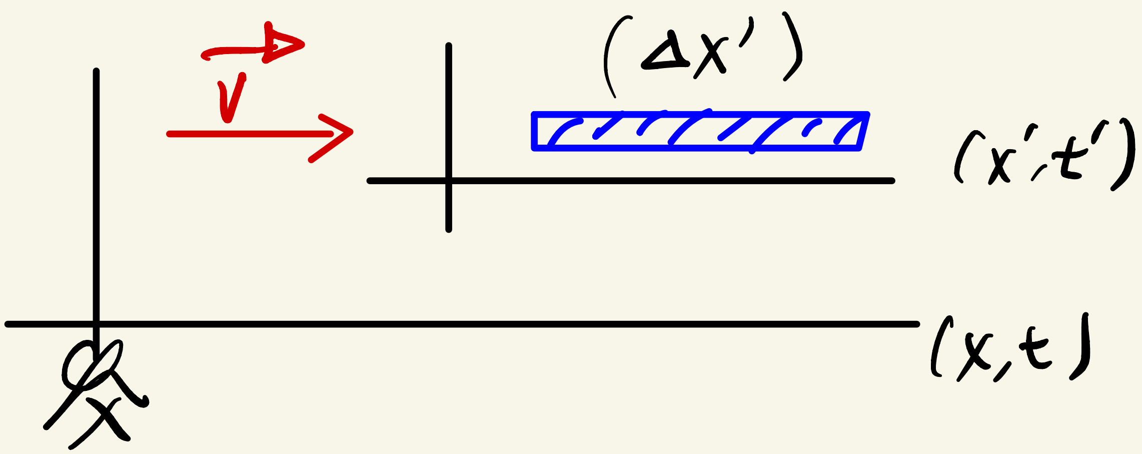
# Length contraction

$$x' = \gamma(x - vt)$$

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

For $\Delta t = 0 \Rightarrow \quad \Delta x' = \gamma \Delta x$

$\Rightarrow \quad \Delta x = \frac{1}{\gamma}\Delta x'$, with $\gamma \geq 1$




From the viewpoint of moving muon, the distance it travels through the atmosphere is shrunk (Lorentz length contraction) by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 40 \quad \text{(in this case)}$$

# Lorentz Velocity transformation

$$\boxed{\begin{aligned} x &= \gamma(x' + vt) \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}} \Rightarrow \boxed{\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) \end{aligned}}$$

Velocities

(divided by $dt'$)

$$u_x \equiv \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'}$$

(along the direction of $\vec{v}$)

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u_y'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$u_z = \frac{u_z'}{\gamma\left(1 + \frac{v}{c^2}u_x'\right)}$$

(Perpendicular to the moving diction of $\vec{v}$)

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The inverse transformations can be readily written down by changing $v$ to $-v$,

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}, \text{ etc.}$$

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Note: No speed can exceed the speed of light ($c = 3\times10^8$ m/sec)

Example: $u_x' = c$, $v = c$

$$\Rightarrow u_x = \frac{u_x' + v}{1 + \frac{v}{c^2}u_x'} = \frac{c + c}{1 + \frac{c}{c^2}c} = c$$

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Velocities

(divided by $dt'$)

$u_x' \equiv \frac{dx'}{dt'}$

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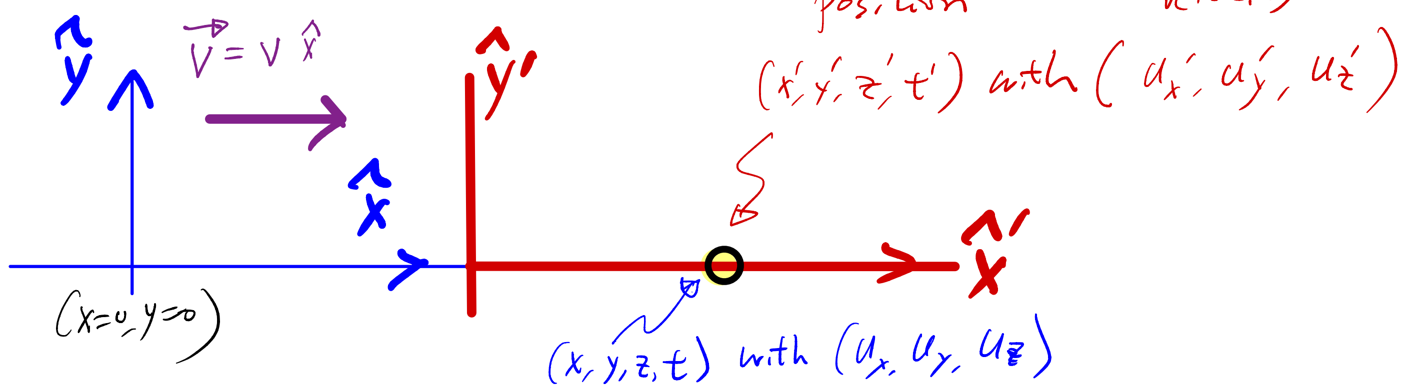
# Lorentz boost along $+\hat{x}$-direction




$$x = \gamma_v \left(x' + \frac{v}{c} t'\right)$$

$$t = \gamma_v \left(t' - \frac{v}{c^2} x^2\right)$$

$$y = y'$$

$$z = z'$$

$$u_x = \frac{u_x' + v}{1 + \frac{v}{c^2} u_x'}$$

$$u_y = \frac{u_y'}{\gamma_v \left(1 + \frac{v}{c^2} u_x'\right)}$$

$$u_z = \frac{u_z}{\gamma_v \left(1 + \frac{v}{c^2} u_x'\right)}$$

$$x' = \gamma_v \left(x - \frac{v}{c} t\right)$$

$$t' = \gamma_v \left(t - \frac{v}{c^2} x\right)$$

$$y' = y$$

$$z' = z$$

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2} u_x}$$

$$u_y' = \frac{u_y}{\gamma_v \left(1 - \frac{v}{c^2} u_x\right)}$$

$$u_z' = \frac{u_z}{\gamma_v \left(1 - \frac{v}{c^2} u_x\right)}$$

with $\boxed{\gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$