Relativistic Kinematics

from

Special Relativity

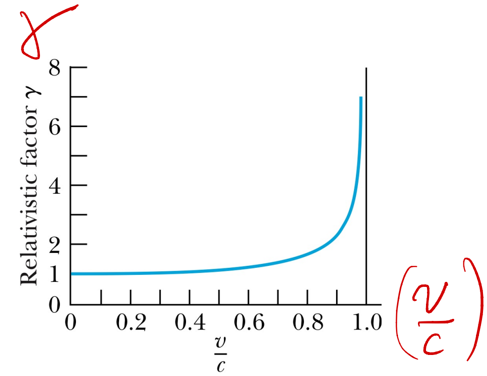
Special Relativity: Relativistic Kinematics

(1) Implication for mass.

— Energy of motion: Increase Mass

$$m = m(v) = m_{rest}\,\gamma, \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

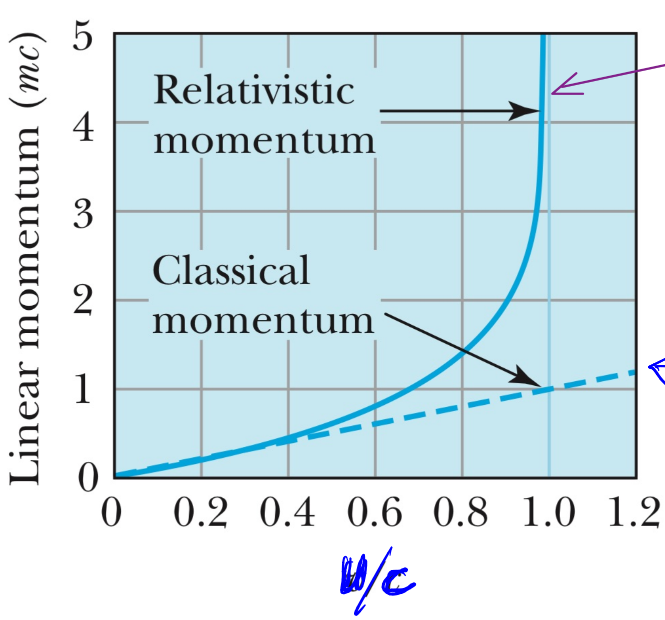
(mass depends on speed)



(2) To respect the conservation laws of linear momentum, we have to use relativistic momentum, defined as

$$\vec{p} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\, m_0 \vec{u}, \quad (\text{where } m_0 \equiv m_{rest})$$

for particle moving with velocity $\vec{u}$.

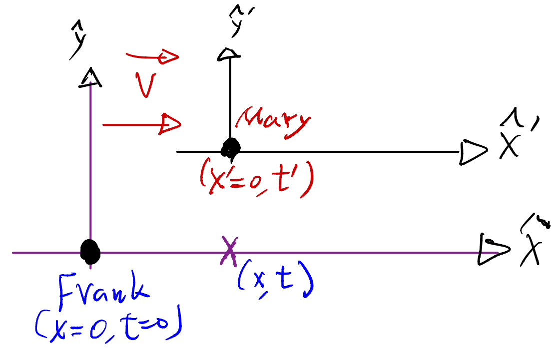


The speed of any particle can not exceed the speed of light. ($u < c$)

$$\vec{p}_{classical} = m_0 \vec{u}$$

(2)' Why using Relativistic momentum?

① Under the Lorentz transformation (Lorentz boost) along the $+\hat{x}$-direction, the transverse velocity is not invariant,



$$u_y \neq u_y'$$

② With the definition of relativistic momentum $\vec{p} = \gamma(u) m_0 \vec{u}$, then the transverse momentum is invariant,

$$P_y = P_y'$$

Note: $\vec{p} = \gamma(u) m_0 \vec{u}$, where $\gamma(u) = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

$\vec{u}$ is the velocity of the particle with its rest mass $m_0 \equiv m(u=0)$,

where the mass

$m \equiv m(u) = \gamma(u) m_0$, with $\gamma(u) = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

$$\boxed{\begin{aligned} u^2 &\equiv \vec{u} \cdot \vec{u} \\ &= u_x^2 + u_y^2 + u_z^2 \\ &\text{for } \vec{u} = (u_x, u_y, u_z) \end{aligned}}$$

Thus, $\vec{p} = m\vec{u}$
$= \gamma(u) m_0 \vec{u}$

with

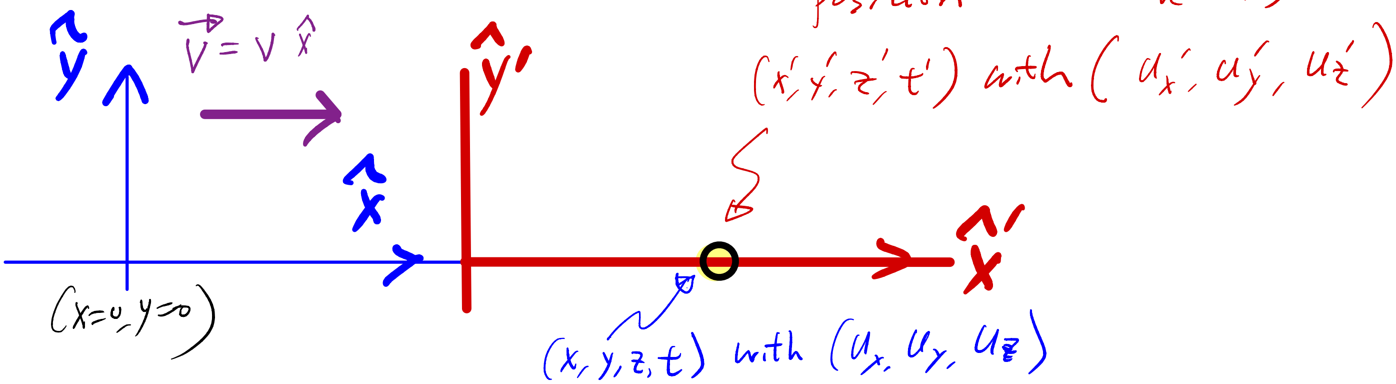
$P_x = \gamma(u) m_0 u_x$,
$P_y = \gamma(u) m_0 u_y$
$P_z = \gamma(u) m_0 u_z$

Note: It is $\gamma(u)$, not $\gamma(u_x)$.

$$\gamma(u) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$\vec{p} = (P_x, P_y, P_z)$

Lorentz boost along $+\hat{x}$-direction



$$x = \gamma_V\left(x' + \frac{V}{c}t'\right)$$

$$t = \gamma_V\left(t' - \frac{V}{c^2}x^2\right)$$

$$y = y'$$

$$z = z'$$

$$u_x = \frac{u_x' + V}{1 + \frac{V}{c^2}u_x'}$$

$$u_y = \frac{u_y'}{\gamma_V\left(1 + \frac{V}{c^2}u_x'\right)}$$

$$u_z = \frac{u_z}{\gamma_V\left(1 + \frac{V}{c^2}u_x'\right)}$$

$$x' = \gamma_V\left(x - \frac{V}{c}t\right)$$

$$t' = \gamma_V\left(t - \frac{V}{c^2}x\right)$$

$$y' = y$$

$$z' = z$$

$$u_x' = \frac{u_x - V}{1 - \frac{V}{c^2}u_x}$$

$$u_y' = \frac{u_y}{\gamma_V\left(1 - \frac{V}{c^2}u_x\right)}$$

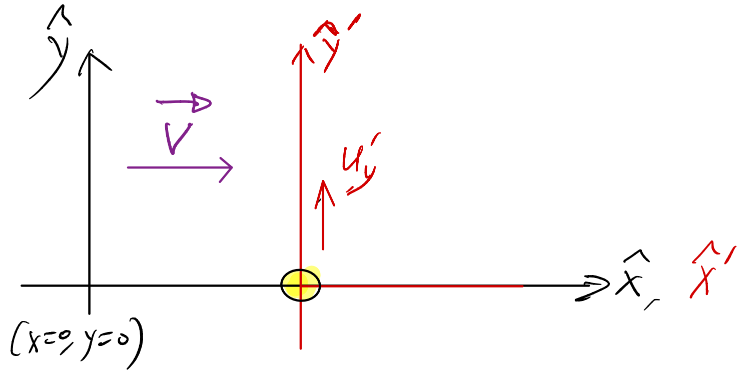
$$u_z' = \frac{u_z}{\gamma_V\left(1 - \frac{V}{c^2}u_x\right)}$$

with $$\boxed{\gamma_V = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}}$$

Let's consider a special case that

$x'=0$, $u_x'=0$, $u_y'\neq 0$, and $y'=0$, $z'=0$, $u_z'=0$.

$\vec{V}=V\hat{x}$
$=V\hat{x}'$



① classical momentum $\quad m_0 u_y \neq m_0 u_y'$

Because

$$u_y = \frac{u_y'}{\gamma_V\left(1+\frac{V}{c^2}u_x'\right)} \overset{(u_x'=0)}{=\!=} \frac{u_y'}{\gamma_V} \neq u_y'$$

② Relativistic momentum $\quad m_0\gamma_u u_y = m_0\gamma_{u'} u_y'$

where

the speed $\quad u \equiv \sqrt{u_x^2+u_y^2+u_z^2} = \sqrt{u_x^2+u_y^2} = \sqrt{V^2+u_y^2}$

$\left(u_x = \frac{V+u_x'}{1+\frac{Vu_x'}{c^2}} = V,\ \text{for } u_x'=0\right)$

$u' \equiv \sqrt{(u_x')^2+(u_y')^2+(u_z')^2} = u_y'$

Hence,

$\left(u_y = \sqrt{1-\frac{V^2}{c^2}}\, u_y'\right)$

$$m_0\gamma_u u_y = \frac{m_0\sqrt{1-\frac{V^2}{c^2}}\,(u_y')}{\sqrt{1-u^2/c^2}} = \frac{m_0(u_y')\sqrt{1-V^2/c^2}}{\sqrt{\left(1-\frac{(u_y')^2}{c^2}\right)\left(1-V^2/c^2\right)}}$$

$$= \frac{m_0(u_y')}{\sqrt{1-(u_y')^2/c^2}} = m_0\gamma_{u'}u_y'$$

(3) The relativistic kinetic energy, K, derived from the definitions of

force $\quad \vec{F} \equiv \dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}\left(\gamma m_0 \vec{u}\right)$, with $\gamma \equiv \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

and the work W_{12} done by a force $\vec{F}$ to move a particle from position 1 to 2 along a path $\vec{s}$,

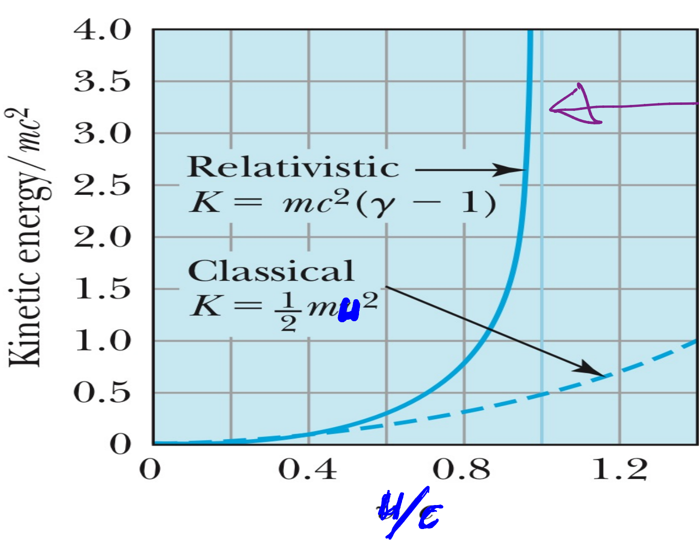
work $\quad W_{12} \equiv \displaystyle\int_1^2 \vec{F}\cdot d\vec{s} = K_2 - K_1$

where, $\quad K = \displaystyle\int \frac{d}{dt}\left(\gamma m_0 \vec{u}\right)\cdot \vec{u}\, dt$

$\Rightarrow$

$$K = \gamma m_0 c^2 - m_0 c^2 = m_0 c^2 (\gamma - 1),$$

with $\gamma = \dfrac{1}{\sqrt{1-\frac{u^2}{c^2}}}$



$\left(\dfrac{u}{c} < 1\right)$

In the limit of $\dfrac{u}{c} \ll 1$,

Relativistic $K \equiv m_0 c^2(\gamma - 1)$

$\simeq \frac{1}{2} m_0 u^2$

(agrees with classical kinetic energy)

(3)' The relativistic kinetic energy

$$K = m_0 \int dt \frac{d}{dt}(\gamma \vec{u}) \cdot \vec{u} = m_0 \int_0^{\gamma u} u \, d(\gamma u)$$

$$= m_0 c^2 (\gamma - 1), \quad \text{for} \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

(integration by parts)

In the limit of $u \ll c$, then $\gamma \simeq 1$,

$$K \overset{u \ll c}{\simeq} m_0 \int_0^u u \, du = \frac{1}{2} m_0 u^2$$

(classical kinetic energy)

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(4) Conservation of Mass-Energy

- Mass and Energy are equivalent
- Mass is another form of Energy
- A particle with no motion still has energy through its mass.

$$E = m(u)c^2 = \gamma m_0 c^2, \quad \text{with } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

- Relationship of Energy and momentum of a particle with rest mass m_0 is

$$E^2 = p^2c^2 + m_0^2c^4$$

$\Rightarrow$ Both $(E^2 - p^2c^2)$ and m_0^2 are invariant quantities under Lorentz transformation from one inertial frame to another.

- For massless particle, such as photon, $m_0 = 0$,

$\Rightarrow$ $$E = pc$$

Using $E = \gamma m_0 c^2$ and $pc = (\gamma m_0 u)c$, we obtain $u = c$ $\Rightarrow$ Massless particle can only travel at the speed of light.

(5) For a particle with rest mass m_0 and velocity $\vec{u}$,

Total energy $E = m_0 \gamma_u c^2$. $\gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$

Kinetic energy $K = m_0 c^2 (\gamma_u - 1)$,

$E = K + m_0 c^2$

$\Longrightarrow$ $K = E - m_0 c^2$,

where $(m_0 c^2)$ is called the rest energy.

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(1) While the relativistic momentum is

$$\vec{p} = m_0 \gamma_u \vec{u},$$

the relativistic kinetic energy

$$K = m_0 c^2 (\gamma_u - 1)$$
$$= m_0 \gamma_u c^2 - m_0 c^2$$

Only this form is correct.

$$K \neq \frac{1}{2m_0} (\vec{p} \cdot \vec{p})$$
$$K \neq \frac{1}{2} m_0 \gamma_u (\vec{u} \cdot \vec{u})$$

(2) As $\frac{u}{c} \ll 1$,

$$K = m_0 c^2 (\gamma_u - 1)$$
$$= m_0 c^2 \left(1 - \frac{u^2}{c^2}\right)^{-1/2} - m_0 c^2$$
$$\overset{\frac{u}{c} \ll 1}{\simeq} m_0 c^2 \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - m_0 c^2$$
$$= \frac{1}{2} m_0 u^2 \leftarrow \text{classical kinetic energy}$$

$\boxed{\text{Note: } (1+x)^a = 1 + ax + \cdots \text{ for } |x| \ll 1}$

$\boxed{\frac{1}{1-x} = 1 + x + \cdots \text{ for } |x| \ll 1}$

(4)' Show that $E^2 = p^2c^2 + m^2c^4$

$$\vec{p} = m_0 \gamma \vec{u}$$

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = u/c$$

$$\beta^2 = 1 - \frac{1}{\gamma^2}$$

$$\begin{aligned}
p^2c^2 &= m_0^2 \gamma^2 u^2 c^2 \\
&= m_0^2 \gamma^2 (c\beta)^2 c^2 \\
&= m_0^2 c^4 \gamma^2 \beta^2 \\
&= m_0^2 c^4 \gamma^2 \left(1 - \frac{1}{\gamma^2}\right) \\
&= m_0^2 c^4 \gamma^2 - m_0^2 c^4 \\
&= E^2 - (m_0 c^2)^2,
\end{aligned}$$

$$\left(E = m_0 \gamma c^2\right)$$

$$\Rightarrow E^2 = p^2c^2 + \underbrace{(m_0 c^2)}_{\text{"rest energy"}}{}^2$$

Summary: (From now on, I will use m, instead of m_0, for rest mass.)

Kinematics for a particle with rest mass m.

classical kinematics

$\vec{p} = m\vec{u}$

$\vec{F} = \dfrac{d\vec{p}}{dt}$

$K = \dfrac{1}{2}mu^2 = \dfrac{p^2}{2m}$

$(u^2 \equiv \vec{u}\cdot\vec{u}\,;\ p^2 = \vec{p}\cdot\vec{p})$

$E = K = \dfrac{p^2}{2m}$

valid for $\dfrac{u}{c} \ll 1$

$\vec{F} = m\dfrac{d\vec{u}}{dt}$

Thus the accelation is

$\dfrac{d\vec{u}}{dt} = \dfrac{1}{m}\vec{F}$

To ensure the law of linear momentum conservation

relativistic kinematics

$\left(\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}\right)$

$\vec{p} = m\gamma\vec{u}$

$\vec{F} = \dfrac{d\vec{p}}{dt}$

$K = mc^2(\gamma - 1)$

$E = K + mc^2$

$E = m\gamma c^2$

$\vec{p} = \dfrac{E}{c^2}\vec{u}$

$E^2 = p^2c^2 + (mc^2)^2$

valid for all u, but $u < c$.

$\vec{F} = \dfrac{d\vec{p}}{dt}$ holds for all v $(<c)$,

but $\vec{F} = m\vec{a} = m\dfrac{d\vec{u}}{dt}$ only holds for $(v \ll c)$.

classical kinematics

$\dfrac{dE}{dt} = \dfrac{dK}{dt} = \vec{F}\cdot\vec{u}$

$\dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}\left(\dfrac{E}{c^2}\vec{u}\right) = \dfrac{E}{c^2}\dfrac{d\vec{u}}{dt} + \dfrac{\vec{u}}{c^2}\dfrac{dE}{dt}$

thus, the acceleration is

$\dfrac{d\vec{u}}{dt} = \dfrac{1}{m\gamma}\left(\vec{F} - \dfrac{\vec{u}}{c^2}(\vec{F}\cdot\vec{u})\right)$

Special Relativity and acceleration

(1)
Space and time intervals as measured by different observers depend only on relative velocity and not on acceleration.

(2)
Mary (on the spaceship) can apply the formulate of special relativity provided that she takes into account the fact that during the periods of acceleration and deceleration she was continuously changing her own (comoving) instantaneous initial frame. For example, right before and after the turning point, the two (comoving) instantaneous initial frames have different velocities (with opposite directions).

Electromagnetic field $F_{\mu\nu}$ under Lorentz transformation

(Lorentz boost)

2. Transformation of $\mathbf{E}$ and $\mathbf{B}$ Under a Boost

For a boost along the **x-axis** with velocity v, the transformation equations are:

In the four spacetime dimensions:

$$X^{\mu} \equiv (ct, x, y, z)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$E'_x = E_x$$

$$E'_y = \gamma(E_y - vB_z)$$

$$E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

where:

- $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

Electrons used to produce medical x rays are accelerated from rest through a potential difference of 25,000 volts before striking a metal target. Calculate the speed of the electrons and determine the error in using the classical kinetic energy result.

Strategy We calculate the speed from the kinetic energy, which we determine both classically and relativistically and then compare the results. In order to determine the correct speed of the electrons, we must use the relativistically correct kinetic energy given by Equation (2.58). The work done to accelerate an electron across a potential difference V is given by qV, where q is the charge of the particle. The work done to accelerate the electron from rest is the final kinetic energy K of the electron.

Solution The kinetic energy is given by

$$K = W = qV = (1.6 \times 10^{-19}\ \mathrm{C})(25 \times 10^{3}\ \mathrm{V})$$
$$= 4.0 \times 10^{-15}\ \mathrm{J}$$

We first determine γ from Equation (2.58) and from that, the speed. We have

$$K = (\gamma - 1)mc^2 \tag{2.60}$$

From this equation, γ is found to be

$$\gamma = 1 + \frac{K}{mc^2} \tag{2.61}$$

The quantity mc^2 for the electron is determined to be

$$mc^2(\text{electron}) = (9.11 \times 10^{-31}\ \mathrm{kg})(3.00 \times 10^{8}\ \mathrm{m/s})^2$$
$$= 8.19 \times 10^{-14}\ \mathrm{J}$$

The relativistic factor is then $\gamma = 1 + [(4.0 \times 10^{-15}\ \mathrm{J})/(8.19 \times 10^{-14}\ \mathrm{J})] = 1.049$. Equation (2.8) can be rearranged to determine β^2 as a function of γ^2, where $\beta = u/c$.

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2} = \frac{(1.049)^2 - 1}{(1.049)^2} = 0.091 \tag{2.62}$$

The value of β is 0.30, and the correct speed, $u = \beta c$, is 0.90×10^8 m/s.

We determine the error in using the classical result by calculating the velocity using the nonrelativistic expression. The nonrelativistic expression is $K = \frac{1}{2}mu^2$, and the speed is given by

$$u = \sqrt{\frac{2(4.0 \times 10^{-15}\ \mathrm{J})}{9.11 \times 10^{-31}\ \mathrm{kg}}}$$
$$= 0.94 \times 10^{8}\ \mathrm{m/s} \quad \text{(nonrelativistic)}$$

The (incorrect) classical speed is about 4% greater than the (correct) relativistic speed. Such an error is significant enough to be important in designing electronic equipment and in making test measurements. Relativistic calculations are particularly important for electrons, because they have such a small mass and are easily accelerated to speeds very close to c.

Note: Total energy is $E = K + mc^2 = m\gamma c^2$

We were taught in introductory physics that the international system of units is preferable when doing calculations in science and engineering. This is generally true, but in modern physics we sometimes use other units that are more convenient for atomic and subatomic scales. In this section we introduce some of those units and demonstrate their practicality through several examples. Recall that the work done in accelerating a charge through a potential difference is given by $W = qV$. For a proton, with charge $e = 1.602 \times 10^{-19}$ C, accelerated across a potential difference of 1 V, the work done is

$$W = (1.602 \times 10^{-19}\underset{\text{C}}{})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$$

In modern physics calculations, the amount of charge being considered is almost always some multiple of the electron charge. Atoms and nuclei all have an exact multiple of the electron charge (or neutral). For example, some charges are proton ($+e$), electron ($-e$), neutron (0), pion ($0, \pm e$), and a singly ionized carbon atom ($+e$). The work done to accelerate the proton across a potential difference of 1 V could also be written as

$$W = (1\ e)(1\ \text{V}) = 1\ \text{eV}$$

where e stands for the electron charge. Thus eV, pronounced "electron volt," is also a unit of energy. It is related to the SI (*Système International*) unit joule by the two previous equations.

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J} \tag{2.74}$$

The eV unit is used more often in modern physics than the SI unit J. The term eV is often used with the SI prefixes where applicable. For example, in atomic and solid state physics, eV itself is mostly used, whereas in nuclear physics MeV (10^6 eV, *mega*-electron-volt) and GeV (10^9 eV, *giga*-electron-volt) are predominant, and in particle physics GeV and TeV (10^{12} eV, *tera*-electron-volt) are used. When we speak of a particle having a certain amount of energy, the common usage is to refer to the kinetic energy. A 6-GeV proton has a *kinetic* energy of 6 GeV, not a *total* energy of 6 GeV. Because the rest energy of a proton is about 1 GeV, this proton would have a total energy of about 7 GeV.