PHY215-11: Dirac equation and Quantum Vacuum fluctuations

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1 Dirac equation

The Dirac equation (1928) was the first *successful* and *consistent* unification of quantum mechanics with special relativity for a real physical particle (the electron).

Before Dirac: the Klein–Gordon equation

The earliest attempt at a relativistic quantum equation was the Klein–Gordon (KG) equation (1926):

$$(\Box + m^2c^2/\hbar^2)\,\phi = 0.$$

It is relativistic and quantum, but it fails as a theory of the electron:

- the probability density is not positive definite;
- it describes spin–0 particles, so it cannot describe the electron (spin 1/2);
- it gives no explanation of the fine structure or spin–orbit coupling;
- it predicts no magnetic moment.

Thus the KG equation was interesting mathematically but not a correct physical theory of electrons.

1.1 Dirac's breakthrough (1928)

Dirac sought a relativistic equation that:

1. is first order in both space and time derivatives;
2. yields a positive-definite probability density;
3. incorporates spin automatically;
4. respects Lorentz invariance.

He introduced the equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where the γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

This equation was revolutionary because it:

- describes spin-1/2 particles via Lorentz spinors;
- predicts the correct electron magnetic moment ($g = 2$);
- explains fine structure and spin–orbit coupling (including the Thomas factor);
- predicts the existence of antimatter (the positron).

It was therefore the first fully consistent union of quantum mechanics and special relativity.

1.2 Non-relativistic expansion of Dirac equation

If you start with the Dirac equation and perform a systematic expansion in powers of v/c or equivalently, in powers of $1/mc^2$ using the Foldy–Wouthuysen transformation, you obtain:

- The non-relativistic Schrödinger equation, plus a series of relativistic correction terms, including
- spin–orbit coupling,
- the Darwin term,
- the relativistic mass correction (p^4 term),
- and the Zeeman spin–magnetic-field term.

This expansion gives the Pauli Hamiltonian and then the fine-structure corrections of the hydrogen atom.

Below is the clean physics summary and the explicit expanded Hamiltonian.

Starting from the Dirac equation in an external electromagnetic field,

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\,\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 + q\phi\right]\psi, \qquad \boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}, \tag{1}$$

one can perform a Foldy–Wouthuysen (FW) transformation to obtain a nonrelativistic expansion in powers of $1/(mc^2)$. Up to order $1/m^2c^2$, the FW Hamiltonian becomes

$$H_{\text{FW}} = mc^2 + \frac{\mathbf{p}^2}{2m} + q\phi - \frac{\mathbf{p}^4}{8m^3c^2} - \frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{q\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}) + \frac{q\hbar^2}{8m^2c^2}(\nabla\cdot\mathbf{E}) + \cdots . \tag{2}$$

Each term has a standard physical interpretation:

- **Relativistic kinetic-energy correction:**

$$H_{\text{kin}} = -\frac{\mathbf{p}^4}{8m^3c^2}. \tag{3}$$

- **Pauli magnetic interaction:**

$$H_{\text{Pauli}} = -\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}, \tag{4}$$

  which encodes the electron's magnetic moment.

- **Spin–orbit coupling:**

$$H_{\text{SO}} = \frac{q\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}). \tag{5}$$

  For a central Coulomb potential $V(r)$, this reduces to the familiar form

$$H_{\text{SO}} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\mathbf{L}\cdot\mathbf{S}. \tag{6}$$

- **Darwin term:**

$$H_D = \frac{q\hbar^2}{8m^2c^2}\,(\nabla\cdot\mathbf{E}), \tag{7}$$

which contributes only for s-wave states in the Coulomb field.

Collecting all contributions, the nonrelativistic expansion of the Dirac equation yields the Schrödinger Hamiltonian, the Pauli Hamiltonian, and the fine-structure corrections:

$$H = \underbrace{\left(\frac{\mathbf{p}^2}{2m} + V\right)}_{\text{Schrödinger}} + \underbrace{\left(-\frac{q\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}\right)}_{\text{Pauli (spin)}} + \underbrace{\left(-\frac{\mathbf{p}^4}{8m^3c^2} + H_{\text{SO}} + H_D\right)}_{\text{fine structure}}. \tag{8}$$

2 Darwin Term in the Dirac Equation

The Dirac equation includes the Darwin term through a non-relativistic expansion of its Hamiltonian, revealing corrections to the Schrödinger equation that account for quantum fluctuations near the nucleus. Here's a step-by-step explanation:

1. **Starting Point: The Dirac Hamiltonian**

$$(i\gamma^\mu \partial_\mu - eA_\mu - m)\psi = 0,$$

where $A_\mu = (\phi, \vec{A})$ is the electromagnetic four-potential. The Hamiltonian form is:

$$H = c\vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta mc^2 + e\phi,$$

with α and β as Dirac matrices.

2. **Non-Relativistic Limit: Foldy-Wouthuysen Transformation**

- Separates positive/negative energy states using v/c expansion.
- Yields corrections including kinetic energy, spin-orbital coupling, and the Darwin term.

3. **Leading Relativistic Corrections**

1. Kinetic Energy Correction: $$-\frac{(\vec{p}^2)^2}{8m^3c^2},$$
2. Spin-Orbital Coupling: $$\frac{e}{4m^2c^2}\vec{\sigma} \cdot (\vec{E} \times \vec{p}),$$
3. Darwin Term: $$\frac{\hbar^2}{8m^2c^2}\nabla^2 V(r),$$

where $V(r) = e\phi$.

4. **Deriving the Darwin Term**

Zitterbewegung (Quantum Fluctuations)

$$\langle V(r) \rangle_{\text{fluctuations}} \approx V(r) + \frac{\hbar^2}{8m^2c^2} \nabla^2 V(r).$$

Relativistic Hamiltonian Expansion

$$H_{\text{Darwin}} = \frac{\hbar^2}{8m^2c^2} \nabla^2 V(r).$$

5. **Physical Interpretation** - Accounts for relativistic quantum corrections near the nucleus. - Explains Lamb shift and hyperfine structure in hydrogen-like atoms.

6. **Mathematical Expression** For $V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$:

$$H_{\text{Darwin}} = \frac{\hbar^2 Ze^2}{8m^2c^2} \cdot \frac{4\pi}{4\pi\epsilon_0} \delta^3(\vec{r}).$$

3 Darwin Term vs. Lamb Shift

The Darwin term and Lamb shift arise from distinct physical origins and produce different numerical results:

1. **Darwin Term**

- **Origin**: Non-relativistic correction from Dirac equation (zitterbewegung and Hamiltonian expansion).

- **Effect**: Modifies electron-nucleus interaction, significant for s states.
- **Numerical Contribution**: Proportional to $\frac{\hbar^2}{8m^2c^2}\nabla^2 V(r)$, smaller than Lamb shift.

2. **Lamb Shift**

- **Origin**: QED effect from vacuum fluctuations and self-energy.
- **Effect**: Splits $2\mathrm{S}_{1/2}$ and $2\mathrm{P}_{1/2}$ levels by $\approx$ 1057 MHz.
- **Numerical Result**: Calculated via QED perturbation theory, depends on α and m.

3. **Key Differences**

Property	Darwin Term	Lamb Shift
Origin	Relativistic QM correction	QED radiative effect
Framework	Dirac equation	Quantum field theory
Magnitude	Smaller (fine structure)	Larger (orders of magnitude)
Physical Cause	Zitterbewegung	Vacuum polarization

4 The Casimir Effect

Predicted by Hendrik Casimir in 1948, this phenomenon arises from quantum vacuum fluctuations of electromagnetic fields, causing neutral metal plates to attract. Here's the mechanism:

1. **Quantum Vacuum Fluctuations** - The vacuum contains virtual particle-antiparticle pairs (e.g., photons) due to Heisenberg's uncertainty principle. - Fluctuations create electromagnetic oscillations with all possible wavelengths.

2. **Boundary Conditions and Mode Restrictions** - Neutral metal plates ($a \sim 10^{-8}\,\mathrm{m}$) act as perfect conductors. - Only standing waves with $\lambda \leq 2a$ exist between plates. - Unlimited modes outside plates create mode number imbalance.

3. **Zero-Point Energy Imbalance** - Each mode has zero-point energy $E = \frac{1}{2}\hbar\omega$. - Lower total energy inside plates due to fewer modes. - Energy difference generates attractive force.

4. **Mathematical Formulation**

$$F = -\frac{\pi^2 \hbar c A}{240 a^4},$$

where: - F: Casimir force (negative = attractive) - A: Plate area - a: Separation distance - Scales as $1/a^4$ (significant at nanoscale).

5. **Physical Interpretation** - Greater vacuum pressure outside plates squeezes them together. - Purely quantum effect (classical physics predicts no force).

6. **Experimental Verification** - 1997: Steven Lamoreaux confirmed force using sphere-plate geometry. - Modern AFM measurements study nanoscale Casimir forces.

7. **Applications and Implications** - **Nanotechnology**: Unintended adhesion in MEMS/NEMS devices. - **Quantum Field Theory**: Validates vacuum energy reality and QED foundations.

Key Takeaway

The Casimir effect demonstrates quantum vacuum fluctuations generate measurable forces between neutral objects, bridging quantum mechanics and relativity. It serves as a cornerstone of QED with implications for fundamental physics and nanotechnology.