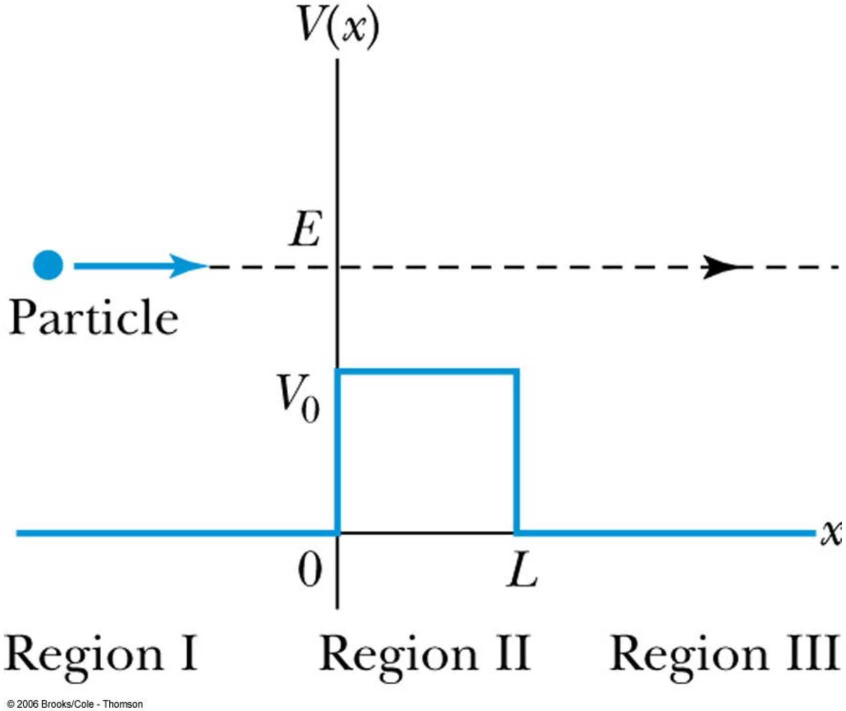
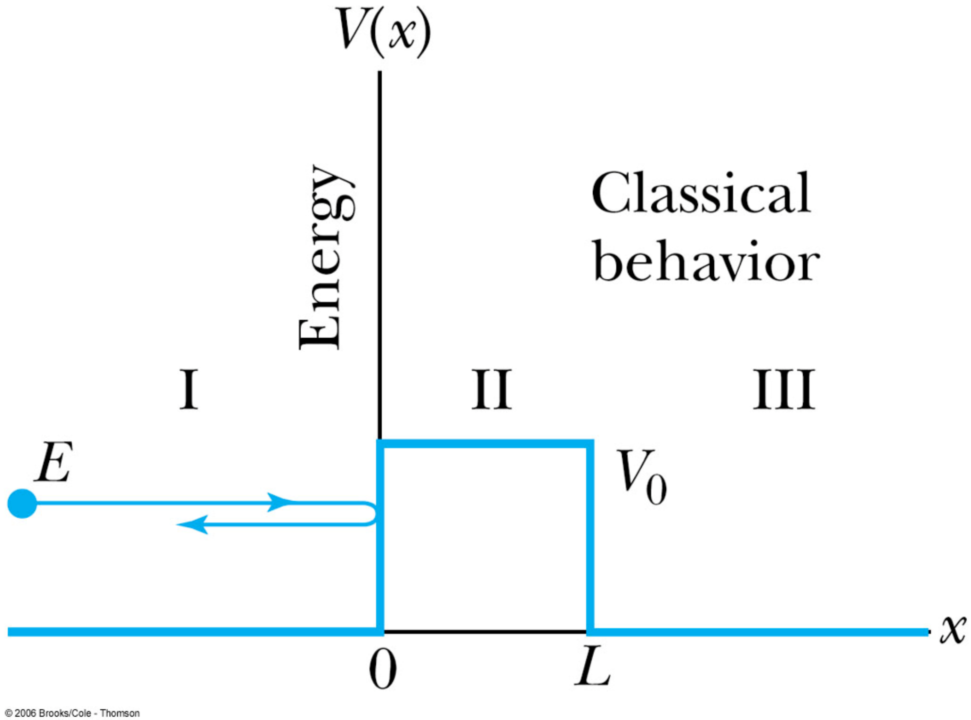
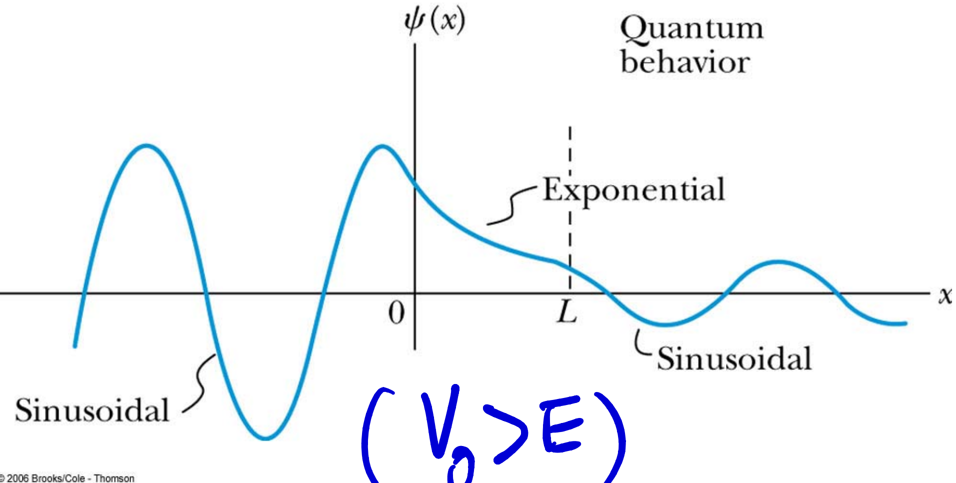
Tunneling





Classically, if a particle approaches a barrier with $E>V_0$, it is transmitted. If $E<V_0$, the classical particle will be reflected but the quantum particle can also tunnel through.

In Regions I,II and III, the Schroedinger equations are



$(V_0 > E)$

$$\frac{d^2u_I}{dx^2} + \frac{2m}{\hbar^2}Eu_I = 0 \quad \leftarrow e^{\pm ikx}$$

$$\frac{d^2u_{II}}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)u_{II} = 0 \quad \leftarrow e^{\pm kx}$$

$$\frac{d^2u_{III}}{dx^2} + \frac{2m}{\hbar^2}Eu_{III} = 0. \quad \leftarrow e^{\pm ikx}$$

If we set

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}},$$

the solutions are familiar:

$$\begin{aligned} u_I(x) &= Ae^{ikx} + Be^{-ikx} \\ u_{II}(x) &= Ce^{-\kappa x} + De^{\kappa x} \\ u_{III}(x) &= Fe^{ikx} + Ge^{-ikx}. \end{aligned}$$

Remember that e^{ikx} moves right and e^{-ikx} moves left. In region III, we can set G=0 because there is only a transmitted wave there.

At this point, we match solutions at $x = 0$ and $x = L$, using, for example,

$$\begin{aligned} u_{II}(L) &= u_{III}(L) \\ \frac{du_{II}(L)}{dx} &= \frac{du_{III}(L)}{dx} \end{aligned}$$

with a similar expression connecting u_I and u_{II} at $x = 0$. The interesting quantity is the ratio $|F|^2/|A|^2$ that measures the tunneling probability. Solving for F, one finds

$$\frac{F}{A} = \frac{2e^{-ika}}{[2\cosh(\kappa a) + i(\kappa/k - k/\kappa)\sinh(\kappa a)]}.$$

The tunneling probability is then

$$\frac{|F|^2}{|A|^2} = \left[1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)}\right]^{-1}.$$

When κL is large, this becomes

$$\frac{|F|^2}{|A|^2} \to 16\,\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2\kappa L}.$$

Suppose an electron is accelerated through a 5 volt potential and strikes a 10 volt barrier of width 0.8 nm. What fraction of the electrons penetrate the barrier?

Here, $L=0.8$ nm, $V_0=10$ eV, $E=5$ eV and κ is

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$= \frac{\sqrt{2(0.511 \times 10^6\,\mathrm{eV/c^2})(10\,\mathrm{eV} - 5\,\mathrm{eV})}}{6.58 \times 10^{-16}\,\mathrm{eV\,s}}$$

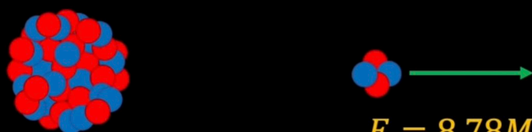
$$\kappa = \frac{3.43 \times 10^{18}\,\mathrm{s}^{-1}}{c} = 1.15 \times 10^{10}\,\mathrm{m}^{-1}.$$

From this, $\kappa L=9.2$, which is large compared to 1. We can then use

$$\frac{|F|^2}{|A|^2} = 16\left(\frac{5\,\mathrm{eV}}{10\,\mathrm{eV}}\right)\left(1 - \left(\frac{5\,\mathrm{eV}}{10\,\mathrm{eV}}\right)\right)e^{-18.4} = 4.1 \times 10^{-8}.$$

Alpha decay of Polonium 212

$$^{212}_{84}Po \rightarrow {}^{208}_{82}Pb + {}^{4}_{2}\alpha$$



$$E = 8.78MeV$$

$$T_{1/2} = 0.3\mu s = 3\times10^{-7}s$$

Tunnelling Probability

$$T \approx \left[\frac{16E(V_0 - E)}{V_0^2}\right] e^{-2\beta a}$$

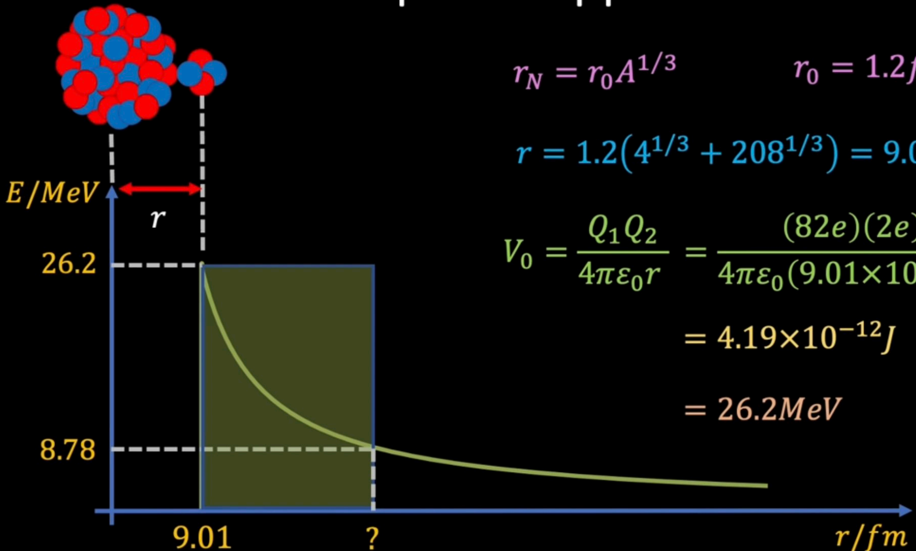
$$T \approx e^{-2\beta a}$$

Tunnelling probability equation

$$T = \frac{|C|^2}{|A|^2} = \frac{C^*C}{A^*A} = \left[\frac{16k^2\beta^2}{(\beta^2 + k^2)^2}\right] e^{-2\beta a}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \beta = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

Simplified approach



$$r_N = r_0 A^{1/3} \qquad r_0 = 1.2fm$$

$$r = 1.2\left(4^{1/3} + 208^{1/3}\right) = 9.01fm$$

$$V_0 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r} = \frac{(82e)(2e)}{4\pi\varepsilon_0(9.01\times10^{-15})}$$

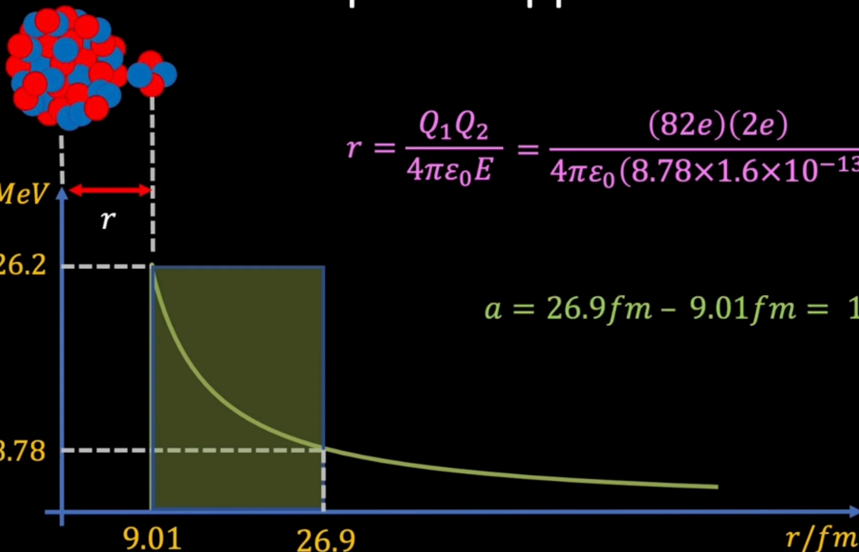
$$= 4.19\times10^{-12}J$$

$$= 26.2MeV$$

Simplified approach

$$r = \frac{Q_1 Q_2}{4\pi\varepsilon_0 E} = \frac{(82e)(2e)}{4\pi\varepsilon_0 (8.78\times1.6\times10^{-13})} = 26.9 fm$$

$$a = 26.9 fm - 9.01 fm = 17.9 fm$$

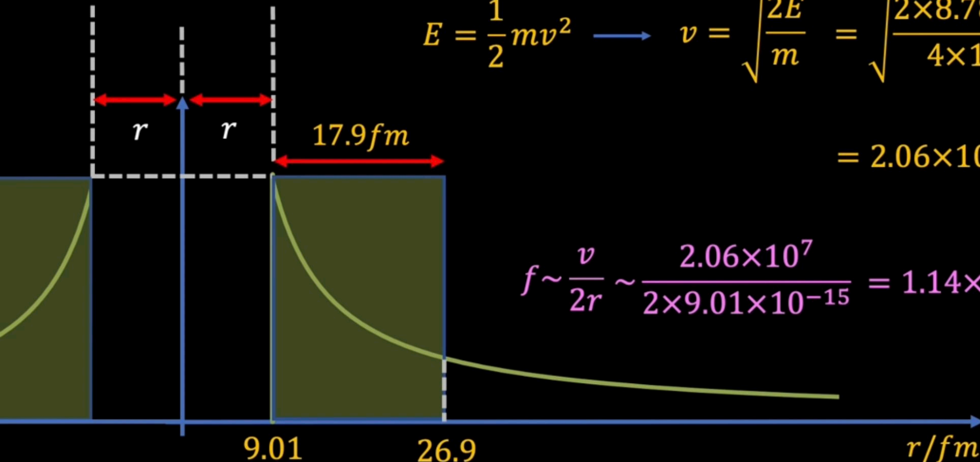


Simplified approach

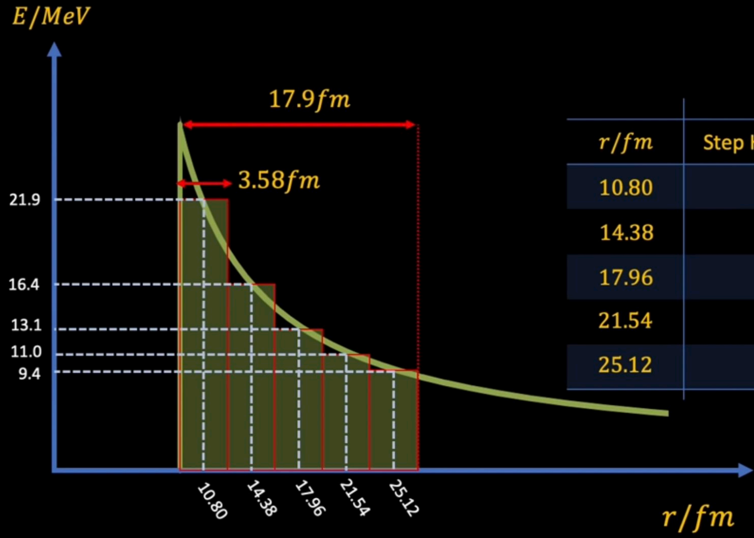
$$E = \frac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2\times 8.78\times 1.6\times 10^{-13}}{4\times 1.67\times 10^{-27}}}$$

$$= 2.06\times 10^{7}\, m/s$$

$$f \sim \frac{v}{2r} \sim \frac{2.06\times 10^{7}}{2\times 9.01\times 10^{-15}} = 1.14\times 10^{21} s^{-1}$$



A better model

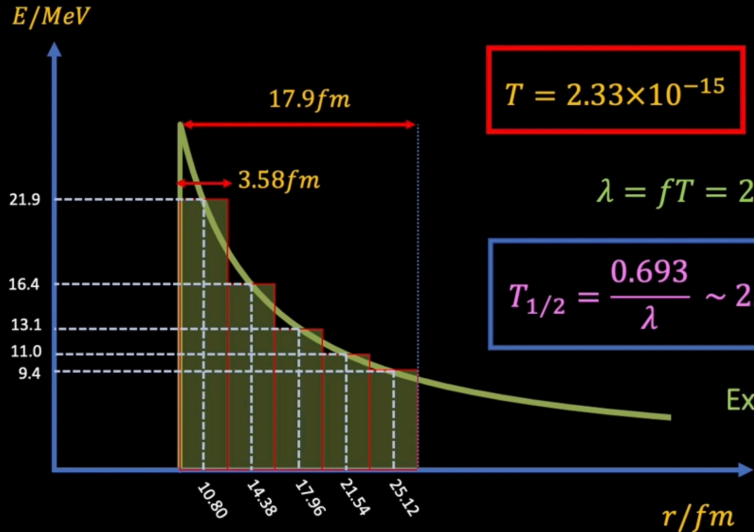


$$T = e^{-2\beta a}$$

r/fm	Step Height/MeV	Tunnelling Probability
10.80	21.9	1.16×10^{-5}
14.38	16.4	1.73×10^{-4}
17.96	13.1	1.47×10^{-3}
21.54	11.0	9.33×10^{-3}
25.12	9.4	8.46×10^{-2}

$$T = 2.33 \times 10^{-15}$$

A better model



$T = 2.33\times10^{-15}$

$f = 1.14\times10^{21}s^{-1}$

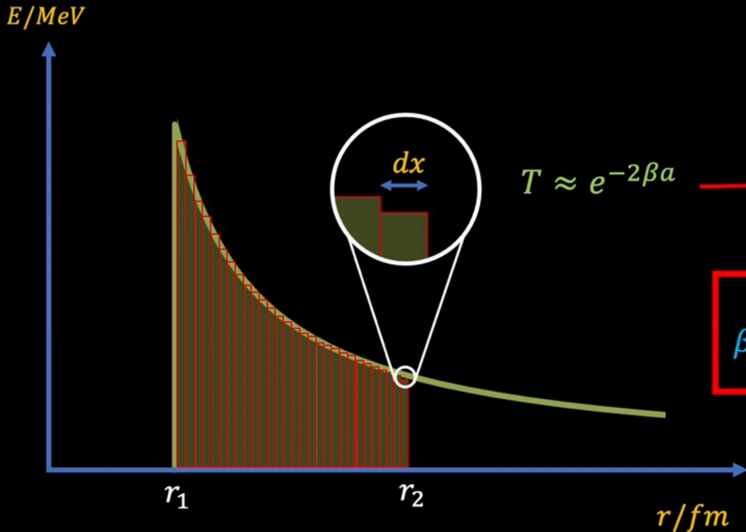
$\lambda = fT = 2.66\times10^{6}s^{-1}$

$T_{1/2} = \frac{0.693}{\lambda} \sim 2.6\times10^{-7} = 0.26\mu s$

Experiment $= 0.3\mu s$

Not bad!

A better model



$$T \approx e^{-2\beta a} \rightarrow T \approx e^{-\int_{r_1}^{r_2} 2\beta(x)dx}$$

$$\beta(x) = \frac{\sqrt{2m(V(x) - E)}}{\hbar}$$